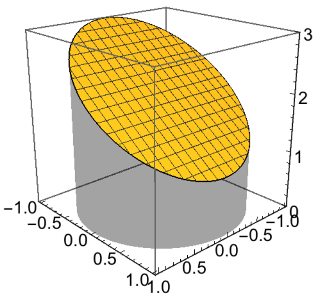
16.8 Stokes' Theorem

Stokes' Theorem states that if S is an oriented piecewise smooth surface bounded by a simple, closed, non-intersecting curve C, then, if $\boldsymbol{F}$ has continuous derivatives

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\,dS = \iint_S (\nabla\times\boldsymbol{F})\cdot d\boldsymbol{S} = \iint_S \operatorname{curl}\boldsymbol{F}\cdot d\boldsymbol{S}$$

where C is traversed in the positive direction.

Example 1 Verify Stokes' Theorem for $\int_C \boldsymbol{F}\cdot d\,\boldsymbol{r}$ where $\boldsymbol{F}(x, y, z) = -y^2\,\boldsymbol{i} + x\,\boldsymbol{j} + z^2\,\boldsymbol{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Example 16.8.1)



Example 2 Verify Stokes' Theorem for $\boldsymbol{F}(x, y, z) = (2x - y)\,\boldsymbol{i} - yz^2\,\boldsymbol{j} - y^2 z\,\boldsymbol{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Example 3 Verify Stokes' Theorem for the line integral along the curve of intersection of the paraboloid $z = 4 - x^2 - y^2$ and the cylinder formed by the functions $y = x^2$ and $y = x$ on the field $F(x, y, z) = y\,\boldsymbol{i} + x\,\boldsymbol{j} + 2z\,\boldsymbol{k}$. (Assume counterclock-wise orientation when viewed from above.)