16.7 Surface Integrals

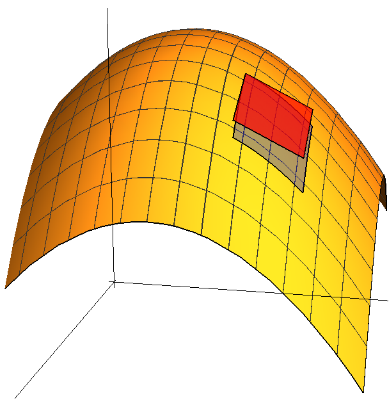
Surface Integrals

The surface integral of a function $f(x, y, z)$ over a surface S defined as $z = g(x, y)$ is:

$$\iint_S f(x, y, z)\, dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(P_{ij}^*\right) \Delta S_{ij} = \iint_D f(x, y, g(x, y)) \sqrt{[g_x(x, y)]^2 + \left[g_y(x, y)\right]^2 + 1}\; dA$$

which is usually written

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y)) \sqrt{\left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2 + 1}\; dA$$



Example 1 Find the surface integral $\iint_S \left(y^2 + 2\, y\, z\right) dS$ where S is the plane $2\, x + y + 2\, z = 6$ in the first octant . Note: the function is in terms of y and z. Alter the above equation to fit this integral.

Center of Mass Moments

If a thin sheet has surface S and density $\rho(x, y, z)$ then the total mass of the sheet is $m = \iint_S \rho(x, y, z)\, dS$ and the center of mass is

$$\overline{x} = \frac{1}{m}\iint_S x\,\rho(x, y, z)\, dS \qquad \overline{y} = \frac{1}{m}\iint_S y\,\rho(x, y, z)\, dS \qquad \overline{z} = \frac{1}{m}\iint_S z\,\rho(x, y, z)\, dS$$

Moments of inertia are defined in a similar fashion.

Example 2 Find the center of mass for a sheet with surface $f(x, y) = 8 - x^2 - y^2$ and density $\delta = 10 - z$ over the square $x : [-2, 2]\ y : [-2, 2]$. Use your calculator for the calculations.

Surface Integrals of Parametric Surfaces

Since the area of a small portion of a parametric surface $\boldsymbol{r}(u, v)$ can be approximated by $\Delta S = \| \boldsymbol{r}_u{}^* \times \boldsymbol{r}_v{}^* \| \, \Delta A$, the surface integral over the parametric surface is:

$$\iint_S f(x, y, z)\, dS = \iint_D f(\boldsymbol{r}(u, v))\, \| \boldsymbol{r}_u \times \boldsymbol{r}_v \| \, dA$$

Example 3 Evaluate $\iint_S (x + y)\, dS$ where S is the portion of the cylinder $y^2 + z^2 = 9$ in the first octant for $0 \le x \le 4$.

The Flux of a Vector Field Through a Surface

We need to now limit ourselves to surfaces that are *orientable* and *smooth*. An orientable surface is one that has two sides (unlike a Möbius strip which only has one side), and *smooth* means the surface has a continuously varying normal vector **n**.

Suppose a fluid has a density δ and a velocity field $\boldsymbol{v}(x, y, z)$, flowing through a surface S, (e.g., a screen or net) then the rate of flow is $\delta\,\mathbf{v}$ (mass per unit time per area). The volume of fluid flowing through S per unit of time is $\delta\,\mathbf{v}\cdot\mathbf{n}\,A(S_{ij})$. Summing these quantities gives

$$\iint_S \delta\,\mathbf{v}\cdot\mathbf{n}\,dS = \iint_S \delta(x,\,y,\,z)\,\mathbf{v}(x,\,y,\,z)\cdot\mathbf{n}(x,\,y,\,z)\,dS$$

which gives the rate of flow through S. If $\boldsymbol{F} = \delta\mathbf{v}$, then we have $\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS$. This is called a *surface integral of* **F** over S, or a *flux integral*.

$$\text{The flux of } \mathbf{F} \text{ across } S = \iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS = \iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$$

Example 4 Calculate the flux integral for $\boldsymbol{F}(x,\,y,\,z) = x\,y\,\boldsymbol{i} + 4\,x^2\,\boldsymbol{j} + y\,z\,\boldsymbol{k}$, and S is the surface $z = x\,e^y$, $0 \le x \le 1$ and $0 \le y \le 1$, with upward orientation.

Notice that if $\boldsymbol{F}(x, y, z) = P(x, y, z)\,\boldsymbol{i} + Q(x, y, z)\,\boldsymbol{j} + R(x, y, z)\,\boldsymbol{k}$, and S is the two-sided surface $z = g(x, y)$ then $\iint\limits_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint\limits_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS = \iint\limits_D \left(-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\right) dA$

Example 5 Evaluate $\iint\limits_S \boldsymbol{F}\cdot d\boldsymbol{S}$ where $\boldsymbol{F}(x, y, z) = 18\,z\,\boldsymbol{i} - 12\,\boldsymbol{j} + 3\,y\,\boldsymbol{k}$ and S is the part of the plane $2\,x + 3\,y + 6\,z = 12$ with upward orientation.

Flux Integrals of Parametric Surfaces

If S is a parametric surface $\boldsymbol{r}(u, v)$ then

$$\iint\limits_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint\limits_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS = \iint\limits_S \boldsymbol{F}\cdot\frac{\boldsymbol{r}_u\times\boldsymbol{r}_v}{|\boldsymbol{r}_u\times\boldsymbol{r}_v|}\,dS = \iint\limits_S \left(\boldsymbol{F}\cdot\frac{\boldsymbol{r}_u\times\boldsymbol{r}_v}{|\boldsymbol{r}_u\times\boldsymbol{r}_v|}\right)\left|\,\boldsymbol{r}_u\times\boldsymbol{r}_v\,\right| dA = \iint\limits_D \boldsymbol{F}\cdot(\boldsymbol{r}_u\times\boldsymbol{r}_v)\,dA$$

Example 6 Evaluate $\iint\limits_S \boldsymbol{F}\cdot d\boldsymbol{S}$ where $\boldsymbol{F}(x, y, z) = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z^4\,\boldsymbol{k}$ and S is the part of the cone $z = \sqrt{x^2 + y^2}$ beneath the plane $z = 1$ with downward orientation.