

## 16.7 Surface Integrals

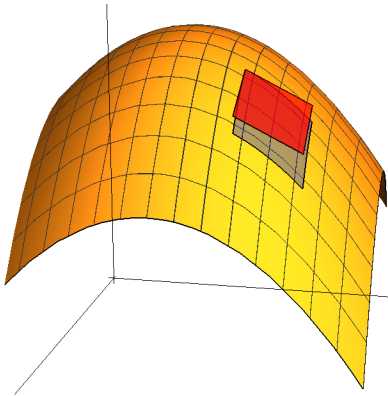
### Surface Integrals

The surface integral of a function  $f(x, y, z)$  over a surface  $S$  defined as  $z = g(x, y)$  is:

$$\iint_S f(x, y, z) dS = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(P_{ij}^*) \Delta S_{ij} = \iint_D f(x, y, g(x, y)) \sqrt{[g_x(x, y)]^2 + [g_y(x, y)]^2 + 1} dA$$

which is usually written

$$\iint_S f(x, y, z) dS = \iint_D f(x, y, g(x, y)) \sqrt{\left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2 + 1} dA$$



**Example 1** Find the surface integral  $\iint_S (y^2 + 2yz) dS$  where  $S$  is the plane  $2x + y + 2z = 6$  in the first octant. Note: the function is in terms of  $y$  and  $z$ . Alter the above equation to fit this integral.

## Center of Mass Moments

If a thin sheet has surface  $S$  and density  $\rho(x, y, z)$  then the total mass of the sheet is  $m = \iint_S \rho(x, y, z) dS$  and the center of mass is

$$\bar{x} = \frac{1}{m} \iint_S x \rho(x, y, z) dS \quad \bar{y} = \frac{1}{m} \iint_S y \rho(x, y, z) dS \quad \bar{z} = \frac{1}{m} \iint_S z \rho(x, y, z) dS$$

Moments of inertia are defined in a similar fashion.

**Example 2** Find the center of mass for a sheet with surface  $f(x, y) = 8 - x^2 - y^2$  and density  $\delta = 10 - z$  over the square  $x : [-2, 2]$   $y : [-2, 2]$ . Use your calculator for the calculations.

## Surface Integrals of Parametric Surfaces

Since the area of a small portion of a parametric surface  $\mathbf{r}(u, v)$  can be approximated by  $\Delta S = \|\mathbf{r}_u^* \times \mathbf{r}_v^*\| \Delta A$ , the surface integral over the parametric surface is:

$$\iint_S f(x, y, z) dS = \iint_D f(\mathbf{r}(u, v)) \|\mathbf{r}_u \times \mathbf{r}_v\| dA$$

**Example 3** Evaluate  $\iint_S (x + y) dS$  where  $S$  is the portion of the cylinder  $y^2 + z^2 = 9$  in the first octant for  $0 \leq x \leq 4$ .

## The Flux of a Vector Field Through a Surface

We need to now limit ourselves to surfaces that are *orientable* and *smooth*. An orientable surface is one that has two sides (unlike a Möbius strip which only has one side), and *smooth* means the surface has a continuously varying normal vector  $\mathbf{n}$ .

Suppose a fluid has a density  $\delta$  and a velocity field  $\mathbf{v}(x, y, z)$ , flowing through a surface  $S$ , (e.g., a screen or net) then the rate of flow is  $\delta \mathbf{v}$  (mass per unit time per area). The volume of fluid flowing through  $S$  per unit of time is  $\delta \mathbf{v} \cdot \mathbf{n} A(S_{ij})$ . Summing these quantities gives

$$\iint_S \delta \mathbf{v} \cdot \mathbf{n} \, dS = \iint_S \delta(x, y, z) \mathbf{v}(x, y, z) \cdot \mathbf{n}(x, y, z) \, dS$$

which gives the rate of flow through  $S$ . If  $\mathbf{F} = \delta \mathbf{v}$ , then we have  $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$ . This is called a *surface integral of  $\mathbf{F}$*  over  $S$ , or a *flux integral*.

The flux of  $\mathbf{F}$  across  $S = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S \mathbf{F} \cdot d\mathbf{S}$

**Example 4** Calculate the flux integral for  $\mathbf{F}(x, y, z) = xy \mathbf{i} + 4x^2 \mathbf{j} + yz \mathbf{k}$ , and  $S$  is the surface  $z = xe^y$ ,  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ , with upward orientation.

Notice that if  $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ , and  $S$  is the two-sided surface  $z = g(x, y)$  then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_D \left( -P \frac{\partial z}{\partial x} - Q \frac{\partial z}{\partial y} + R \right) dA$$

**Example 5** Evaluate  $\iint_S \mathbf{F} \cdot d\mathbf{S}$  where  $\mathbf{F}(x, y, z) = 18z\mathbf{i} - 12\mathbf{j} + 3y\mathbf{k}$  and  $S$  is the part of the plane  $2x + 3y + 6z = 12$  with upward orientation.

## Flux Integrals of Parametric Surfaces

If  $S$  is a parametric surface  $\mathbf{r}(u, v)$  then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \, dS = \iint_S \left( \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \right) \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA$$

**Example 6** Evaluate  $\iint_S \mathbf{F} \cdot d\mathbf{S}$  where  $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z^4\mathbf{k}$  and  $S$  is the part of the cone  $z = \sqrt{x^2 + y^2}$  beneath the plane  $z = 1$  with downward orientation.