16.6 Parametric Surfaces

A curve in $\mathbb{R}^3$ can be parametrized using a singe variable, e.g., $\boldsymbol{r}(t) = f(t)\,\boldsymbol{i} + g(t)\,\boldsymbol{j} + h(t)\,\boldsymbol{k}$ where the vector $\boldsymbol{r}$ is a position vector of a point on the curve. A surface in $\mathbb{R}^3$, $z = f(x, y)$, is parametrized using two parameters, u and v:

$$\boldsymbol{r}(u, v) = x(u, v)\,\boldsymbol{i} + y(u, v)\,\boldsymbol{j} + z(u, v)\,\boldsymbol{k}$$

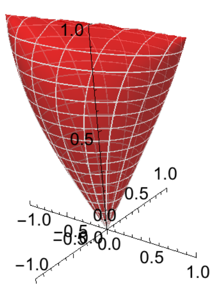
Example 1 Graph the function $\boldsymbol{r}(u, v) = (u - v^2)\,\boldsymbol{i} + (u - u^3)\,\boldsymbol{j} + (v - 2u^2)\,\boldsymbol{k}$ for $-2 \le v \le 2$ and $-2 \le v \le 2$.

Example 2 Describe the graph of $\mathbf{r}(u, v) = 2\cos(u)\,\mathbf{i} + v\,\mathbf{j} + (v^2 + 2\sin(u))\,\mathbf{k}$.

Example 3 Find a parametrization for the surface $y = x\,z^2 - z$ and describe the grid curves.

Example 4 Find a parametrization for the surface of the sphere $x^2 + y^2 + z^2 = 16$ that lies between the planes $z = -2$ and $z = 2$.

Example 5 The *"eccentric conic"* is defined by $\boldsymbol{r}(u, v) = u\cos(v)\,\boldsymbol{i} + u\sqrt{1 - u^2}\,\sin(v)\,\boldsymbol{j} + u\,\boldsymbol{k}$, where $0 \le v \le 2\pi$ and $0 \le u \le 1$. A cross section at height c of the cone is an ellipse with eccentricity c and semi-major axis length c, hence, c and u are the eccentricity.



Manipulate of the *Eccentric Conic*:

Tangent Planes to Parametric Surfaces

The normal vector, **n**, for a tangent plane to the parametric surface $\boldsymbol{r}(u, v)$ is found with $\boldsymbol{n} = \boldsymbol{r}_u \times \boldsymbol{r}_v$ for (u^*, v^*), where

$$\boldsymbol{r}_u = \frac{\partial x}{\partial u}\boldsymbol{i} + \frac{\partial y}{\partial u}\boldsymbol{j} + \frac{\partial z}{\partial u}\boldsymbol{k} \quad \text{and} \quad \boldsymbol{r}_v = \frac{\partial x}{\partial v}\boldsymbol{i} + \frac{\partial y}{\partial v}\boldsymbol{j} + \frac{\partial z}{\partial v}\boldsymbol{k}$$

Example 6 Find the tangent plane to the surface in *Example 1* at the point $\boldsymbol{r}(1, 2)$

Example 7 Find the tangent plane to the eccentric conic when $u = \frac{\pi}{4}$ and $v = 0.5$.

Surface Area of Parametric Surfaces

The surface area of a parametric surface is given by:

$$\text{SA} = \iint_D \| \mathbf{r}_u \times \mathbf{r}_v \| \, dA$$

Example 8 Show that this gives the surface area of the function $z = f(x, y)$ by parametrizing the surface as $\boldsymbol{r}(x, y) = x\,\boldsymbol{i} + y\,\boldsymbol{j} + f(x, y)\,\boldsymbol{k}$.

Example 9 Find the surface area of the part of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.

Example 10 Find the surface area of the eccentric conic. (May need to use *Mathematica*).