16.5 Curl and Divergence

The Curl of a Vector Field

If $\boldsymbol{F} = P\,\boldsymbol{i} + Q\,\boldsymbol{j} + R\,\boldsymbol{k}$ is a vector field in $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the **curl** of **F** is given by

$$\text{curl}\,\boldsymbol{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}$$

An easier way to remember, or derive, the curl is to redefine "del",

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}$$

Note, $\nabla f = \text{grad}(f)$. However, we can also use "del" as a cross product with $\boldsymbol{F}$:

Example 1 Find and simplify $\nabla \times \boldsymbol{F}$.

What is the curl? If **F** denotes the velocity field for a fluid, the curl **F** gives the *direction* of the axis about which the fluid rotates (curls) most rapidly, and $|\text{curl}\,\boldsymbol{F}|$ is a measure of the speed of this rotation. The direction of the rotation is according to the right-hand rule.

Example 2 Let $\boldsymbol{F}(x, y, z) = x^2 y z\,\boldsymbol{i} + 3xyz^3\,\boldsymbol{j} + (x^2 - z^2)\,\boldsymbol{k}$. Find curl $\boldsymbol{F}$ at the point (1, 1, 1).

Example 3 Let $f(x, y, z)$ be a function in $\mathbb{R}^3$ with continuous second partial derivatives. Find curl(∇f).

The Divergence of a Vector Field

If $\boldsymbol{F} = P\,\boldsymbol{i} + Q\,\boldsymbol{j} + R\,\boldsymbol{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of F** is the function of three variables given by

$$\operatorname{div}\boldsymbol{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Example 4 Notice that div $\boldsymbol{F}$ is a scalar field. Rewrite div $\boldsymbol{F}$ using ∇.

What is divergence: If **F** denotes the velocity of a fluid, then div **F** at a point **p** measures the tendency of that fluid to diverge away from **p** (div $\boldsymbol{F} > 0$) or accumulate toward **p** (div $\boldsymbol{F} < 0$).

Example 5 Let $\boldsymbol{F}(x, y, z) = x^2\,y\,z\,\boldsymbol{i} + 3\,x\,y\,z^3\,\boldsymbol{j} + \left(x^2 - z^2\right)\boldsymbol{k}$. Find div $\boldsymbol{F}$ at (1, 1, 1), (1, 1, 0), and (0, 1, 1).

Example 6 Let $\boldsymbol{F} = P\,\boldsymbol{i} + Q\,\boldsymbol{j} + R\,\boldsymbol{k}$ be a vector field in $\mathbb{R}^3$ with continuous partial derivatives, find div curl $\boldsymbol{F}$

Example 7 Let f be a scalar field and **F** a vector field. Which of the following expressions are scalar fields, vector fields, or meaningless?

(a) div f	(b) grad f	(c) curl $\boldsymbol{F}$	(d) div(grad f)
(e) curl(grad f)	(f) grad(div $\boldsymbol{F}$)	(g) curl(curl $\boldsymbol{F}$)	(h) div(div $\boldsymbol{F}$)
(i) grad(grad f)	(j) div(curl(grad f))	(k) curl(div(grad f))	

Vector Forms of Green's Theorem

Example 7 We can write a vector in $\mathbb{R}^2$ as a vector in $\mathbb{R}^3$ where the third component is $0\,\boldsymbol{k}$, i.e., $\boldsymbol{F}(x, y) = P(x, y)\,\boldsymbol{i} + Q(x, y)\,\boldsymbol{j} + 0\,\boldsymbol{k}$. Find $\nabla \times \boldsymbol{F}$, and rewrite the line integral $\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}$ in terms of this expression.

Another form of Green's Theorem in the plane uses the unit tangent to the curve C, $\boldsymbol{T} = \frac{dx}{ds}\,\boldsymbol{i} + \frac{dy}{ds}\,\boldsymbol{j}$ and the unit normal vector $\boldsymbol{n} = \frac{dy}{ds}\,\boldsymbol{i} - \frac{dx}{ds}\,\boldsymbol{j}$. Given the vector field $\boldsymbol{F}(x, y) = P(x, y)\,\boldsymbol{i} + Q(x, y)\,\boldsymbol{j}$, then

$$\begin{aligned}\oint_C \boldsymbol{F}\cdot\boldsymbol{n}\,ds &= \oint_C (P\,\boldsymbol{i} + Q\,\boldsymbol{j})\cdot\left(\frac{dy}{ds}\,\boldsymbol{i} - \frac{dx}{ds}\,\boldsymbol{j}\right)ds \\ &= \oint_C (-Q\,dx + P\,dy) \\ &= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)dA\end{aligned}$$