16.4 Green's Theorem

Green's Theorem relates the line integral of a simple closed curve of a continuous vector field to the integral of a function over the enclosed area.

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region enclosed by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Example 1 For the force field $\boldsymbol{F} = x^2 y\,\boldsymbol{i} + x\,y\,\boldsymbol{j}$ and the closed simple piecewise path from (0, 0) to (1, 1) along $y = x^2$ and then back to (0, 0) along $y = x$,

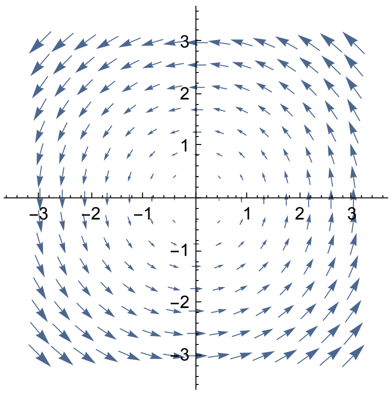
a) Calculate the work using the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$

b) and with $\oint P\,dx + Q\,dy$

c) and with Green's theorem.

Example 2 Use Green's theorem to evaluate $\oint (2x+y^2)\,dx+(x^2+2y)\,dy$ where C is the closed curve formed by $y=0$, $x=2$, and $y=\frac{x^3}{4}$.

Example 3 For the force field $\boldsymbol{F}=\frac{-1}{2}y\,\boldsymbol{i}+\frac{1}{2}x\,\boldsymbol{j}$ and a simple closed curve C, use Green's theorem to interpret the line integral $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$.



Example 4 Use Green's Theorem to find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Green's Theorem Over Non-Simple Regions

Green's theorem holds true even if the region D has one or more holes.