16.3 Fundamental Theorem of Line Integrals

Recall the Fundamental Theorem of Calculus part 2 which states:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

where f' is continuous on $[a, b]$. The *Fundamental Theorem for Line Integrals* is a similar statement.

Theorem Let C be a smooth curve $\boldsymbol{r}(t)$ for $a \le t \le b$. Let f be a differentiable function in two or three variables whose gradient vector ∇f is continuous on C. Then,

$$\int_C \nabla f \cdot d\boldsymbol{r} = f(\boldsymbol{r}(b)) - f(\boldsymbol{r}(a))$$

Q1: What type of vector field is ∇f?
Q2: What does that mean to calculate a line integral on such a field from point P_1 to P_2?

Example 1 Given $f(x, y, z) = 2x^2 + yz$ evaluate $\int_C \nabla f \cdot d\boldsymbol{r}$ from the point $(1, 3, 2)$ to $(4, 1, 3)$.

Since a line integral on a conservative vector field depends only on the values at the end points, any two paths C_1 and C_2 with the same endpoints will give the same value for the line integral. We say *line integrals of conservative vector fields are independent of path.*

Example 2 For the vector field $\boldsymbol{F}(x, y) = 4xy\,\boldsymbol{i} + 2x^2\,\boldsymbol{j}$ (from example 16.2.5), evaluate the line integral from (0, 0) to (2, 4) along $y = x^2$, then back to (0, 0) linearly.

Theorem $\int_C \boldsymbol{F} \cdot dr$ is independent of path in D if and only if $\int_C \boldsymbol{F} \cdot dr = 0$ for every closed path C in D.

Theorem Suppose $\boldsymbol{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ is independent of path in *D,* then $\boldsymbol{F}$ is a conservative vector field on *D.*

So, the question still remains, how can you tell if a vector field $\boldsymbol{F}$ is conservative or not? Suppose $\boldsymbol{F} = M\,\boldsymbol{i} + N\,\boldsymbol{j}$ is conservative, where *M and N* have first-order partial derivatives. Then there is a function *f*, called a *potential function*, such that $\nabla f = \boldsymbol{F}$; or $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$. Using Clairaut's theorem $\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}$, a conservative vector field must have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$:

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial N}{\partial x}$$

Example 3 Determine whether $\boldsymbol{F}(x, y) = (4x^3 + 9x^2y^2)\,\boldsymbol{i} + (6x^3y + 6y^5)\,\boldsymbol{j}$ is a conservative vector field, and if so, find a potential function f.

Example 4 For the vector field in example 3, calculate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where C is any path from (0, 0) to (1, 2).

Example 5 Show that $\boldsymbol{F}(x, y, z) = (e^x \cos(y) + yz)\,\boldsymbol{i} + (xz - e^x \sin(y))\,\boldsymbol{j} + xy\,\boldsymbol{k}$ is a conservative vector field, and find the potential function $f(x, y, z)$ such that $\nabla f = \boldsymbol{F}$.

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