16.2 Line Integrals

So far in calculus we've looked at integrals evaluated on an interval $[a, b]$, an area R in $\mathbb{R}^2$, and even a solid region in $\mathbb{R}^3$:

$$\int_a^b f(x)\,dx \qquad \iint_R f(x, y)\,dA \qquad \iiint_S f(x, y, z)\,dV$$

Consider now, a curve in $\mathbb{R}^2$, e.g., $y = g(x)$, or parametrically $\boldsymbol{r}(t) = \langle x(t), y(t)\rangle$. Integrating a function over the curve is called a **line integral**, and is written

$$\int_C f(x, y)\,ds \text{ or } \int_C f(x, y, z)\,ds \text{ in } \mathbb{R}^3$$

as long as the curve C is continuous and smooth, i.e., $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$.

As we can see, a line integral is "with respect to the arc length" of the curve C, which means we can write a line integral as

$$\int_C f(x, y)\,ds = \int_a^b f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt\,,$$

or in $\mathbb{R}^3$

$$\int_C f(x, y, z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt.$$

Note, that if $f(x, y, z) = 1$ then the line integral gives the arc length of the curve C, or $\int_C 1\,ds = \int_a^b \|\boldsymbol{r}'(t)\|\,dt$.

Example 1 Evaluate the line integral for $f(x, y) = x$ on the curve $y = x^2$ from the point $(0, 0)$ to $(2, 4)$.

The line integral in example 1 can be interpreted as the area under the "curtain" of f along the curve C:

Example 2 Find a piecewise continuous function for the curve from $(0, 0)$ to $(4, 0)$ along the x-axis, vertically to $(4, 2)$ and then back to $(0, 0)$ along the path $y = \sqrt{x}$.

Example 3 Set up the line integral for $f(x, y) = x + y$ on the curve C in example 2.

Line Integrals of Vector Fields

Suppose we want to calculate the work in moving a particle along a curve $\boldsymbol{r}(t)$ through a force field $\boldsymbol{F}(x, y)$. We know that work = force × distance, so a small incremental amount of work ΔW is the amount of force $\boldsymbol{F}$ in the direction of $\boldsymbol{r}$. This amount is found by calculating the scalar projection of the force vector $\boldsymbol{F}$ onto the unit tangent vector $\boldsymbol{T}$ over a small length of curve Δs: $\Delta w \approx \boldsymbol{F} \cdot \boldsymbol{T} \Delta s$. From this we see

$$\begin{aligned} \boldsymbol{F} \cdot \boldsymbol{T} ds &= \boldsymbol{F} \cdot \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} \|\boldsymbol{r}'(t)\| \, dt \\ &= \boldsymbol{F} \cdot \boldsymbol{r}'(t) \, dt \\ &= \boldsymbol{F} \cdot d\boldsymbol{r} \end{aligned}$$

DEFINITION Let $\boldsymbol{F}$ be a continuous vector field defined on a smooth curve C given by $\boldsymbol{r}(t)$ where $a \le t \le b$. The **line integral** of $\boldsymbol{F}$ along C is given by

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \boldsymbol{F} \cdot \boldsymbol{T} \, ds = \int_a^b \boldsymbol{F}(x(t), y(t), z(t)) \cdot \boldsymbol{r}'(t) \, dt$$

Example 4 Find the work done by the force field $\boldsymbol{F}(x, y, z) = -\frac{1}{2} x \, \boldsymbol{i} - \frac{1}{2} y \, \boldsymbol{j} + \frac{1}{4} \boldsymbol{k}$ on a particle that moves along the helix given by $\boldsymbol{r}(t) = \cos(t) \, \boldsymbol{i} + \sin(t) \, \boldsymbol{j} + t \, \boldsymbol{k}$ from the point (1, 0, 0) to $(-1, 0, 3\pi)$.

Example 5 Find the work done by the force field $\boldsymbol{F}(x, y) = 4xy \, \boldsymbol{i} + 2x^2 \, \boldsymbol{j}$ in moving a particle from the point (0, 0) to (1, 1) along any path. (Does the path matter?)

Differential Form of a Line Integral

Example 6 Let $\boldsymbol{F}(x, y) = M \, \boldsymbol{i} + N \, \boldsymbol{j} = 2x \, \boldsymbol{i} + 3xy \, \boldsymbol{j}$, and C is the curve from (1, 1) to (3, 9) along $y = x^2$. Calculate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ and $\int_C M \, dx + N \, dy$.