16.1 Vector Fields

Definition 1: Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function **F** that assigns to each point (x, y) in D a two-dimensional vector $\boldsymbol{F}(x, y)$:

$$\boldsymbol{F}(x, y) = P(x, y)\,\boldsymbol{i} + Q(x, y)\,\boldsymbol{j} = \langle P(x, y),\ Q(x, y)\rangle$$

Definition 2: Let E be a set in $\mathbb{R}^3$ (a volumetric region). A **vector field on** $\mathbb{R}^3$ is a function **F** that assigns to each point (x, y, z) in E a three-dimensional vector $\boldsymbol{F}(x, y, z)$:

$$\boldsymbol{F}(x, y, z) = P(x, y, z)\,\boldsymbol{i} + Q(x, y, z)\,\boldsymbol{j} + R(x, y, z)\,\boldsymbol{k} = \langle P(x, y),\ Q(x, y),\ R(x, y, z)\rangle$$

Example 1 Examples of vector fields on $\mathbb{R}^2$.

$\boldsymbol{F}(x, y) = \langle 2x - y,\ x + y\rangle$

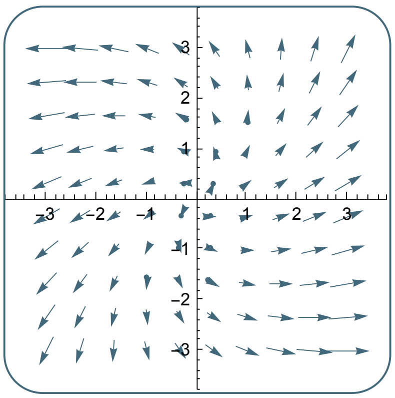

Figure 1

$\boldsymbol{F}(x, y) = \left\langle \frac{-1}{2}x,\ \frac{1}{2}y\right\rangle$

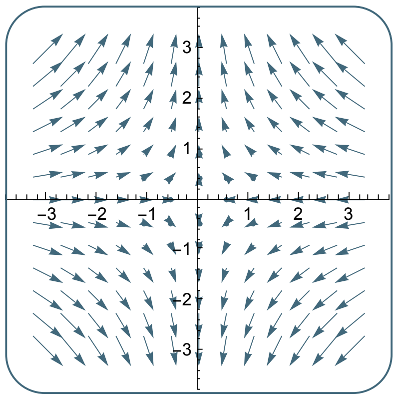

Figure 2

$\boldsymbol{F}(x, y) = \left\langle \cos(2xy),\ \sin\left(x^2 + y\right) - x\right\rangle$

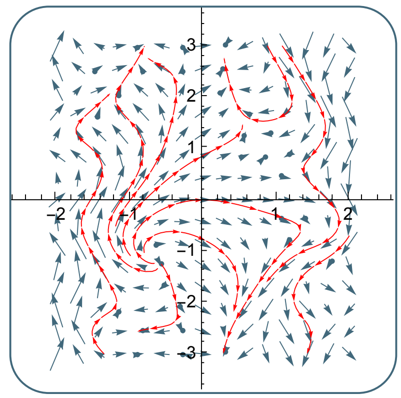

Figure 3

Figure 3 also show several "stream lines" flowing through the vector field.

A basic command to create a vector field in *Mathematica:*

```
VectorPlot[{x^2 - y, x + y + 1}, {x, -3, 3}, {y, -3, 3}]
```

Example 2 An example of the vector field $\boldsymbol{F}(x, y, z) = y\,\boldsymbol{i} + z\,\boldsymbol{j} + x^2\,\boldsymbol{k}$ on $\mathbb{R}^3$:

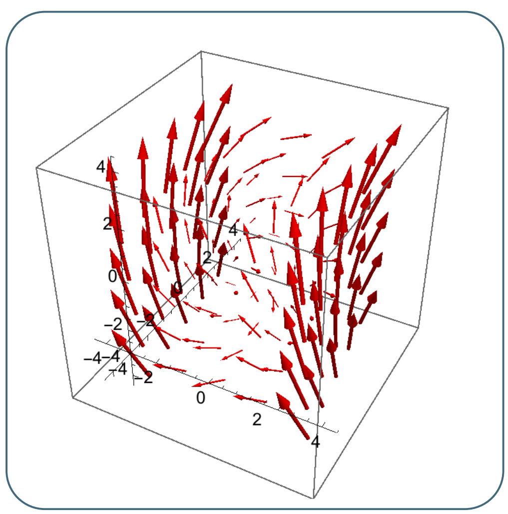

Figure 4

$\boldsymbol{F}(x, y, z) = \left\langle y,\ z,\ x^2\right\rangle$

Mathematica command:

```
VectorPlot3D[{y, z, x^2}, {x, -3, 3}, {y, -3, 3}, {z, -3, 3}, VectorStyle → {{Red, "Arrow3D"}}]
```

Gradient Field

Recall the gradient of a function in several variables creates a gradient vector, or more precisely, a vector field called a gradient field:

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = f_x\, \boldsymbol{i} + f_y\, \boldsymbol{j}$$

Example 3 A plot of the gradient field for $f(x, y) = \sqrt{x + y^2}$, along with level curves, and streamlines.

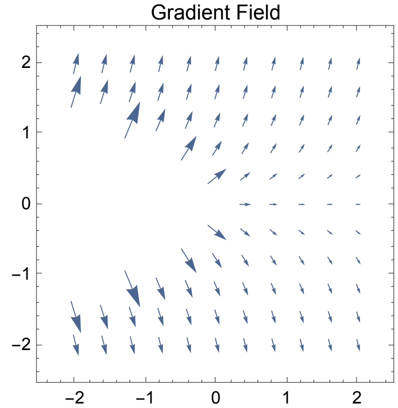

Figure 5

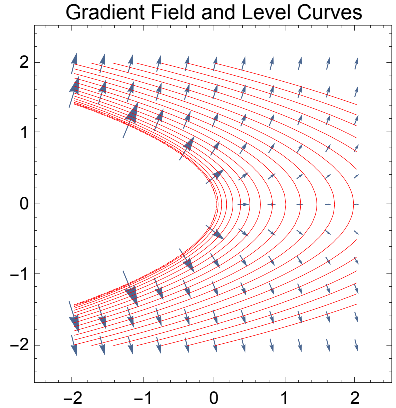

Figure 6

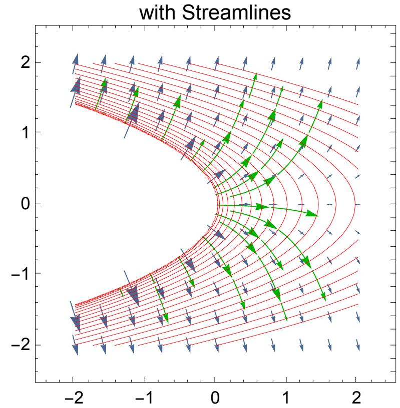

Figure 7

If a vector field $\boldsymbol{F}$ is a gradient field of a scalar function, that is, $\boldsymbol{F} = \nabla f$, we call the vector field *conservative.* Also, the scalar function f is called a *potential function* for $\boldsymbol{F}$. We'll look into this in later sections.

Example 4 Find and make a graph of the conservative vector field given the potential function $f(x, y) = x\, y + y$.