15.9 Change of Variable - The Jacobian

The formulas for converting from rectangular coordinates to polar, cylindrical, and spherical coordinates, are specific examples of *change of variables*. Even in single variable calculus when integrating using substitution illustrates a much more general result.

Suppose we wish to change the variables from (x, y) to (u, v), and that the equations $x = g(u, v)$ and $y = h(u, v)$ relate the old variables (x, y) to the new variables (u, v). We can think of this as a transformation T, such that $(u, v) \underset{\rightarrow}{T} (x, y)$, (or the image S in u and v, is the image R in x and y) and, if T is one-to-one then $(x, y) \underset{\rightarrow}{T^{-1}} (u, v)$.

Example 1 Find the image of the unit square in quadrant I (u and v coordinates) if T is the transformation $x = u - v^2$ and $y = 2uv$.

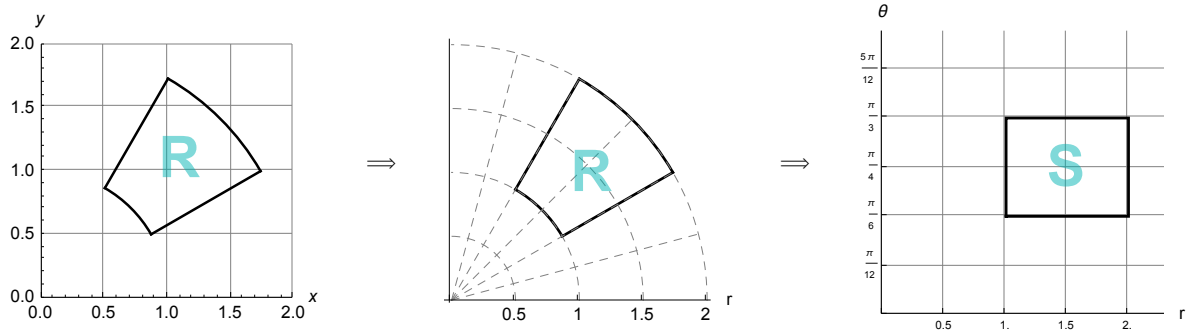
Define the function $J(u, v)$, called the **Jacobian** (the Jacobian Determinant), by

$$J(u, v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Then, we can write the integration for the change in variables from (x, y) in a region R to (u, v) in region S as

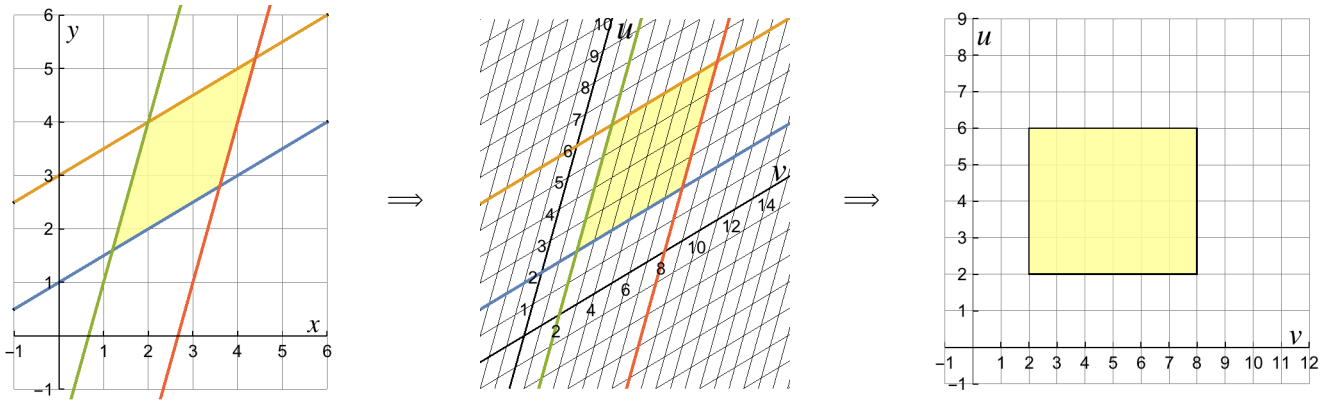
$$\iint_R f(x, y)\, dA = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv = \iint_S f(g(u, v), h(u, v)) \left| J(u, v) \right| du\, dv$$

Geometrically, we are transforming the region R to S, and for equal areas we have: $R = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| S$.



Example 2 Find the Jacobian for the transformation from rectangular to polar coordinates.

Example 3 Evaluate $\iint_R 25\,x\,y\,dA$ where R is the region bounded by: $y = \frac{1}{2}x + 1$, $y = \frac{1}{2}x + 3$, $y = 3x - 2$, and $y = 3x - 8$. Note: the intersections of the lines are $\left(\frac{6}{5}, \frac{8}{5}\right)$, $\left(\frac{18}{5}, \frac{14}{5}\right)$, (2, 4), and $\left(\frac{22}{5}, \frac{26}{5}\right)$.



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Example 4 Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$ by making the change of variables $x = a\,u$, $y = b\,v$, and $z = c\,w$. Also find the moment of inertia about the z-axis assuming it has a constant density k.

Example 5 Find the value of the integral $\iint_R (x - y)\,dA$ where R is the triangular region with corner points (0, 0), (3, 1) and (2, 2).