15.8 Cylindrical and Spherical

Cylindrical Coordinates

Cylindrical coordinates are an extension of polar coordinates; a point in $\mathbb{R}^3$ with rectangular coordinates $R(x, y, z)$ can be converted into cylindrical coordinates $C(r, \theta, z)$ with $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = z$ (See figure 1). This means a small "cylindrical rectangle" will have volume $r\,\Delta\theta\,\Delta r\,\Delta z$, (see figure 2) , which leads to the triple integral conversion from rectangular to cylindrical coordinates

$$\iiint_S f(x, y, z)\,dV = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\int_{z_1}^{z_2} f(r\cos(\theta), r\sin(\theta), z)\,r\,dz\,dr\,d\theta$$

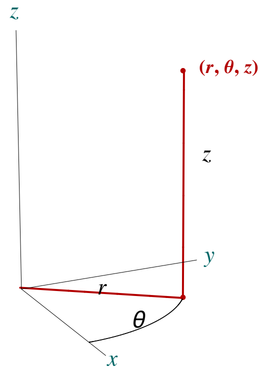


Figure 1

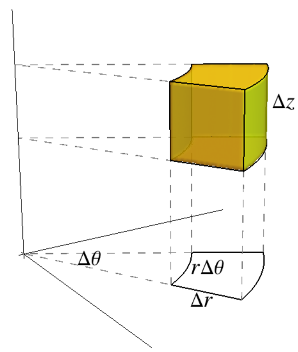


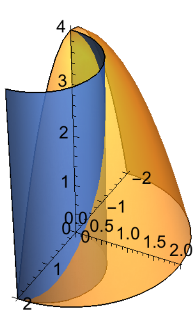
Figure 2

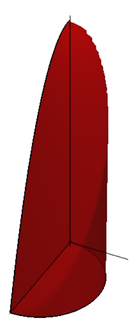
In addition, the z integration limits can be functions of r and θ, and the limits for r can be functions of θ, which gives the iterated integrals in cylindrical coordinates:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,dz\,dy\,dx = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos(\theta), r\sin(\theta), z)\,r\,dz\,dr\,d\theta$$

Example 1 Find the mass and center of mass of a circular cylinder with radius a and height h, if the density is proportional to the height above the xy-plane.

Example 2 Find the volume of the solid bounded above by the paraboloid $z = 4 - x^2 - y^2$, below by $z = 0$, and laterally by $y = 0$ and the cylinder $x^2 + y^2 = 2x$.





Spherical Coordinates

Recall spherical coordinates are (ρ, θ, ϕ) and the conversion from rectangular coordinates (x, y, z) are

$$x = \rho \sin(\phi)\cos(\theta), \qquad y = \rho \sin(\phi)\sin(\theta), \qquad \text{and } z = \rho\cos(\phi)$$

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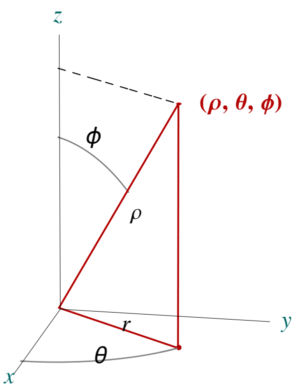

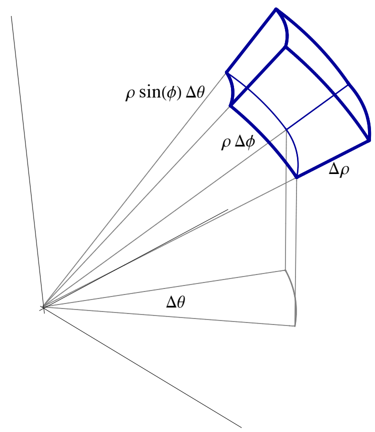

It can be shown that the volume of a spherical wedge for $\Delta\rho$, $\Delta\theta$, and $\Delta\phi$ is $\rho^2 \sin(\phi)\,\Delta\rho\,\Delta\theta\,\Delta\phi$. Therefore, we have

$$\iiint_S f(x, y, z)\,dV = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\,\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$

Example 3 Find the mass of a solid sphere of radius a if its density δ is proportional to the distance from the center.