15.7 Triple Integrals

Let B be the rectangular region in $\mathbb{R}^3$: $a \le x \le b$, $c \le y \le d$, and $s \le z \le t$, and $f(x, y, z)$ be a continuous function, then

$$\iiint_S f(x, y, z)\, dV = \int_a^b \int_c^d \int_s^t f(x, y, z)\, dz\, dy\, dx$$

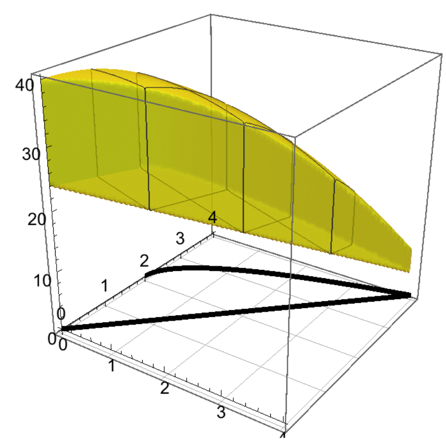
Example 1 Evaluate $\iiint_B x^2 z\, dz\, dy\, dx$ where B is the box $B = \{(x, y, z) \mid 0 \le x \le 2,\ 2 \le y \le 5,\ 1 \le z \le 3\}$.

A Triple Integral over a General Region

Suppose we have a region S in $\mathbb{R}^3$ bounded by two surfaces $z = u_1(x, y)$ and $z = u_2(x, y)$, where $u_1(x, y) \le u_2(x, y)$. Also, let S have projection D in the xy-plane bounded by two curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \le g_2(x)$. Finally, let $a \le x \le b$. The iterated integral of $f(x, y, z)$ over the region B is given by:

$$\iiint_S f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dx$$

Example 2 Let $f(x, y, z) = x^2 y + z$. Set up the integral to evaluate $\iiint_B f(x, y, z)\, dV$ where B is bounded by the surfaces $3x + 2y + z = 24$ and $z = 40 - x^2 - y^2$, over the domain bounded by $y = \sqrt{x} + 2$ and $y = x$.



Example 3 The temperature of the tetrahedron with vertices (0, 0, 0), (2, 0, 0), (0, 4, 0), and (0, 0, 8) is given by $T(x, y, z) = -2z + 16$. Find the total heat content (enthalpy) of the tetrahedron.

Example 4 Graph the solid represented by $\int_0^4 \int_0^{\sqrt{x}} \int_0^{4-x} f(x, y, z)\, dz\, dy\, dx$, and rewrite the integral in the other 5 orders of integration.

Moments, Centers of Mass, and Moments of Inertia

$$\text{mass} = m = \iiint_B \rho(x, y, z)\, dV$$

$$M_{yz} = \iiint_B x\, \rho(x, y, z)\, dV \qquad M_{xz} = \iiint_B y\, \rho(x, y, z)\, dV \qquad M_{xy} = \iiint_B z\, \rho(x, y, z)\, dV$$

$$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

$$I_x = \iiint_B (y^2 + z^2)\, \rho(x, y, z)\, dV \qquad I_y = \iiint_B (x^2 + z^2)\, \rho(x, y, z)\, dV \qquad I_z = \iiint_B (x^2 + y^2)\, \rho(x, y, z)\, dV$$

Example 5 For the solid in example 2, suppose the density of the object is $\rho(x, y, z) = x^2 + y^2 + z$, set up the integrals to find the center of mass, the moment of inertia about the z-axis, and the radius of gyration about the z-axis of an equivalent point mass. Use *Mathematica* to evaluate.