15.5 Applications: Centers of Mass and Inertia

In Calc II we looked at centers of mass of lamina with constant density. Suppose you have a lamina with region D and varying density $\rho(x, y)$, where ρ is continuous on D. The total mass of the lamina can be found by calculating the mass for a small rectangular region R_{ij} with area ΔA, summing up all such regions, and taking the limit as $\Delta A \to 0$:

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho\left(x_{ij}^*, y_{ij}^*\right) \Delta A = \iint_D \rho(x, y)\, dA$$

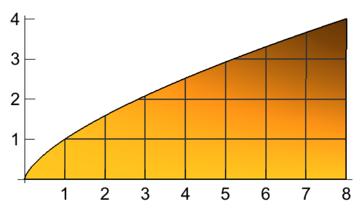
Moments and Center of Mass

The formulas for the moments and centers of mass are now similar to the formulas derived previously:

$$M_x = \iint_D y\, \rho(x, y)\, dA \qquad M_y = \iint_D x\, \rho(x, y)\, dA$$

$$\text{and:} \quad \overline{x} = \frac{M_y}{m} \qquad \overline{y} = \frac{M_x}{m}$$

Example 1 A lamina with density $\rho(x, y) = x\, y^2$ is bounded by the x-axis, the line $x = 8$, and the curve $y = x^{2/3}$. Find the center of mass.



Example 2 A lamina in the shape of a quarter circle of radius a has density proportional to the distance from the center of the circle. Find the center of mass. Hint: use polar coordinates.

Moment of Inertia

Recall from physics that the kinetic energy, KE, of a particle with mass m and velocity v moving in a straight line is $KE = \frac{1}{2} m v^2$. If instead, the particle rotates about an axis with an angular velocity of ω radians per unit of time, its linear velocity is $v = \omega r$, where r is the radius of its circular path. This gives

$$KE = \frac{1}{2} m (\omega r)^2$$
$$= \frac{1}{2} (m r^2) \omega^2$$

The expression $m r^2$ is called the **moment of inertia** (sometimes called the second moment) and is denoted I. If a system is made up of several point masses with distances $r_1,\ r_2,\ \ldots r_n$, from a line L, the moment of inertia of the system about L is

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots m_n r_n^2 = \sum_{k=1}^{n} m_k r_k^2$$

Now, consider a lamina with density $\rho(x, y)$ on a region R in the xy-plane, the moment of inertia about the x-, y-, and z-axes are given by

$$I_x = \iint_R y^2 \rho(x, y)\, dA \qquad I_y = \iint_R x^2 \rho(x, y)\, dA \qquad I_z = \iint_R (x^2 + y^2) \rho(x, y)\, dA$$

Notice that $I_z = I_x + I_y$, and also that $x^2 + y^2 = r^2$ where r is the distance of each point-mass from the origin, or pole. I_z is often referred as the **polar moment of inertia**. All of these moments are also known as *second moments*.

Example 3 Find the moments of inertia for the object in example 1.

Radius of Gyration

Consider a region R rotating about a line L, and wanting to replace the region with a point mass m with the same moment of inertia with respect to the line. How far from the line should the point mass be located? If we call the distance from the point mass to the line $\bar{\bar{r}}$, then the moment of inertia is $I = m \bar{\bar{r}}^2$, which means, the distance from the line is $\bar{\bar{r}} = \sqrt{\frac{I}{m}}$. This is called the **radius of gyration** of the system. Thus, the kinetic energy of the system is: $\text{KE} = \frac{1}{2} m\, \omega^2 \bar{\bar{r}}^2$. In particular, the radius of gyration with respect to the y-axis is $\bar{\bar{x}} = \sqrt{\frac{I_y}{m}}$, and with respect to the x-axis is $\bar{\bar{y}} = \sqrt{\frac{I_x}{m}}$.

Example 4 Find the radius of gyration with respect to the y-axis, x-axis, and the pole for the object in example 1. Find the kinetic energy (when revolved around the pole) of the system in example 1 assuming SI units, and $\omega = 1$ rev/sec.