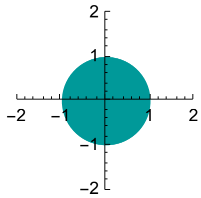
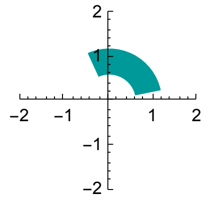
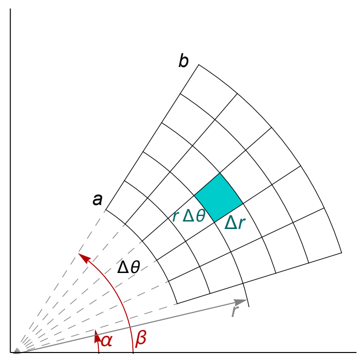
15.4 Double Integrals in Polar Coordinates

Suppose we need to evaluate the integral $\iint_R f(x, y)\, dA$ where the region R is described by a circular region about the origin, i.e.,





We can see that $x^2 + y^2 = r^2$ is an underlying factor, as is the polar Rectangle $a \le r \le b$ and $\alpha \le \theta \le \beta$. To see how to convert the integral from rectangular coordinates to polar coordinates we subdivide the region R, and look at *polar rectangular* subregions:

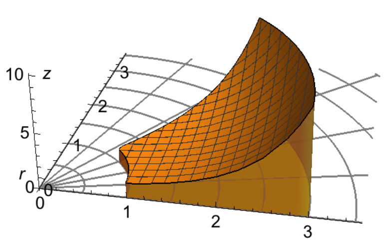


As $\Delta\theta \to 0$ and $\Delta r \to 0$ the area of the polar rectangular region $\Delta A \to dA = r\, dr\, d\theta$. This gives us the following rectangular to polar conversion:

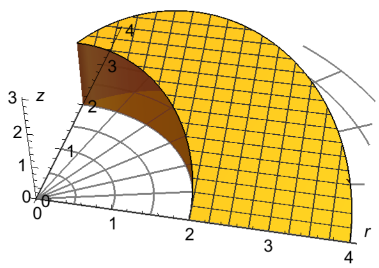
Suppose $f(x, y)$ is a continuous function defined on a polar rectangle $a \le r \le b$ and $\alpha \le \theta \le \beta$ where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

Example 1 Calculate the volume of the solid under the surface $z = e^{(x^2+y^2)/4}$ above the polar region $R = \left\{(r, \theta) : 1 \le r \le 3,\ 0 \le \theta \le \frac{\pi}{4}\right\}$.



Example 2 Given the region R in the first quadrant inside the cardioid $r = 2(1 + \cos(\theta))$ and outside the circle $r = 2$, evaluate $\iint_R y \, dA$.



Example 3 Calculate the volume of the solid under the surface $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2y$.