15.3 Volume of Solids with Non-Rectangular Bases

Suppose a solid has a non-rectangular base bounded by two functions $y_1 = g_1(x)$ and $y_2 = g_2(x)$ where $g_1(x) < g_2(x)$ for all $x \in [a, b]$, e.g., the base may have the shape below in Figure 1, and a height $z = f(x, y)$. A possible solid is show in Figure 2.

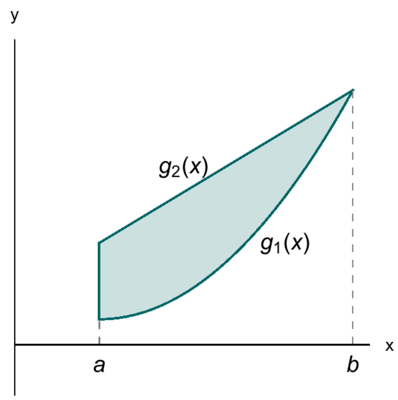


Figure 1

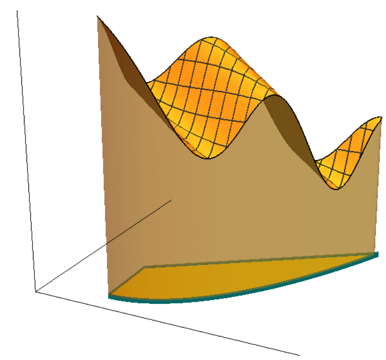
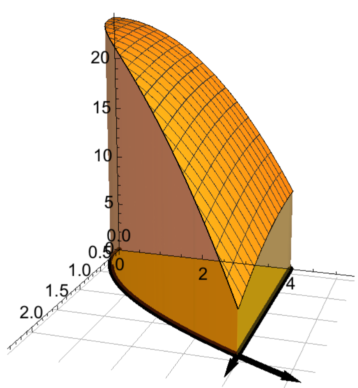


Figure 2

Using the tried and true concept of "slice, approximate, and integrate", we can slice the base region parallel to the y-axis from g_1 to g_2. These slices run in the x direction from a to b, and the height at each point is $f(x, y)$. This leads to the expression:

$$\text{Volume} = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Example 1 Find the volume of the solid that has a top surface given by $f(x, y) = 24 - x^2 - y^2$, and base in the first quadrant bounded by $y = x^2$ and $y = 4$.



Example 2 Graph the base region of the solid given by $\int_0^2 \int_0^{x^3} f(x, y)\, dy\, dx$, and change the order of integration.

Example 3 Find the value of $\int_0^4 \int_{x/2}^2 e^{y^2}\, dy\, dx$.

Example 4 Evaluate: $\int_0^{\pi/2} \int_y^{\pi/2} \frac{\sin(x)}{x}\, dx\, dy$

Example 5

(a) Use a double integral to calculate the area of the region bounded by $y = x^2$, $y = 2 - x$, and the y-axis.

(b) Calculate the mass of the lamina in (a) if the density of the lamina is $\delta(x, y) = x + 2y \ \text{gr}/\text{cm}^2$.