15.2 Iterated Integrals

The double integral $\iint_R f(x, y)\, dA$ for $x \in [a, b]$ and $y \in [c, d]$ can be written in the form $\int_a^b \int_c^d f(x, y)\, dy\, dx$. To evaluate an iterated integral, think of it as

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_{x=a}^{b} \left[\int_{y=c}^{d} f(x, y)\, dy \right] dx$$

That is, integrate with respect to y (thinking of x as a constant) and then integrate with respect to x.

Example 1 Evaluate the integral: $\int_2^4 \int_{-1}^2 \left(4x^2 y + 3y^2 x\right) dy\, dx$

Example 2 Evaluate the integral: $\int_0^1 \int_0^2 \left(x^2 y - x^2\right) dx\, dy$ and $\int_0^2 \int_0^1 \left(x^2 y - x^2\right) dy\, dx$.

Fubini's Theorem

If f is continuous on the rectangle $x \in [a, b]$ and $y \in [c, d]$, then $\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$.

Example 3 Evaluate: $\int_0^1 \int_0^2 x\, e^{x y}\, dx\, dy$ as simply as possible.

Example 4 Find the exact volume under the surface $f(x, y) = 8 - x^2 + 2y - y^2$ on the region $R = \{(x, y) \mid 0 \le x \le 2,\ 1 \le y \le 3\}$. (This is the volume from 15.1 notes we approximated to 25.5.)

Example 5 Show that $\int_a^b \int_c^d f(x)\, g(y)\, dy\, dx = \int_a^b f(x)\, dx \cdot \int_c^d g(y)\, dy$. Evaluate example 2 using this method.

Example 6 Calculate the volume below the surface $f(x, y) = x^2 + e^y$ and above the plane $-2x + 3y + z = -5$ on the region $[0, 2] \times [0, 2]$.