15.1 Double Integrals over Rectangular Regions

Recall the area under a function $f(x) \geq 0$ on the interval $[a, b]$ can be approximated by a Riemann Sum:

As $\Delta x \to 0$ (or $n \to \infty$), we wrote the Riemann sum as

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x)\,\Delta x = \int_a^b f(x)\,dx$$

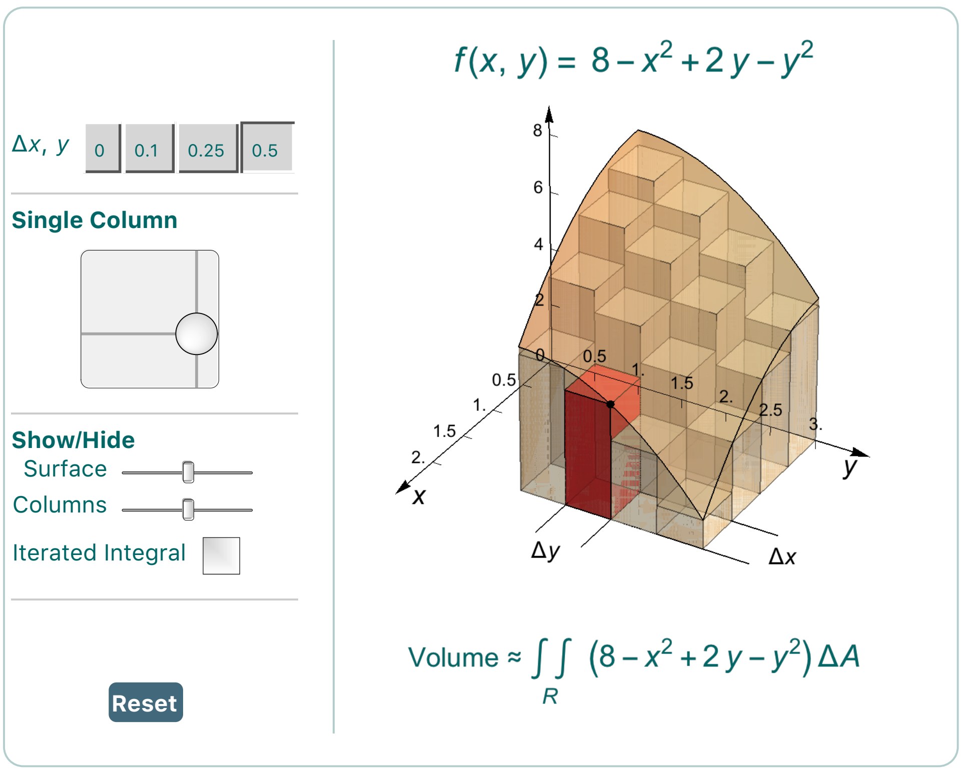
Volumes and Double Integrals

We can also approximate the volume under a multivariable function $f(x, y) \geq 0$ defined on a rectangular region R: $x \in [a, b]$ and $y \in [c, d]$, in a similar fashion using a double summation.

$$\text{volume} \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}, y_{ij}\right) \Delta A$$

Taking the limit as both n and $m \to \infty$ we get the exact volume and represent the volume by a *double integral*

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}, y_{ij}\right) \Delta A = \iint_R f(x, y)\,dA$$



Example 1 Evaluate: $\iint_R \sqrt{4 - x^2}\,dA$ where $R = \{(x, y) \mid 0 \leq x \leq 2,\ 0 \leq y \leq 4\}$.

Example 2 Approximate the volume of the solid given above using four subdivisions in both the x and y direction using the midpoint of each subdivision. That is, approximate $\iint_R \left(8 - x^2 + 2y - y^2\right) dA$ where $R = \{(x, y) \mid 0 \le x \le 2,\ 1 \le y \le 3\}$.

Average Value of a Function

We can also calculate the average value of a function in two variables just as we did with a single variable function:

Single Variable: $f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$ **Two-Variables:** $f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x, y)\,dA$

Example 3 Estimate the average value of the function in example 1.