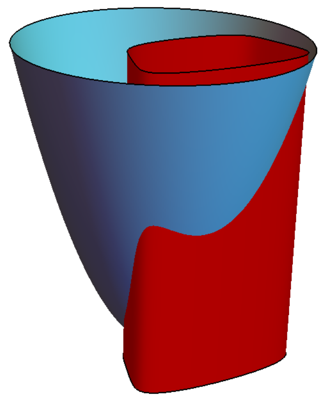
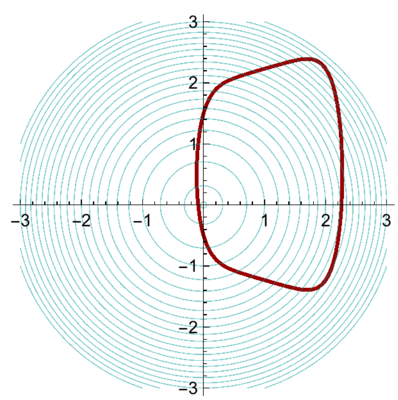
14.8 Constrained Extrema Using Lagrange Multipliers

Consider finding the local extrema of the function $f(x, y) = x^2 + y^2$ with the constraint that x and y needs to satisfy the equation $(x - 1)^6 - x + (y + 0.5)^2 = 2$. Graphically we can see there are multiple local extrema:



We can write the constraint as the function $g(x, y) = 0$, and can get a better idea of how to find the extrema by looking the level curves of f and g:



You can see that the *gradient* vectors of the function f and g are multiples of each other at the extrema.

Symbolically, the local extrema occur when

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

The value λ is called a ***Lagrange Multiplier***.

Method of Lagrange Multipliers

To find the extreme values of a function $f(x, y)$ subject to the constraint $g(x, y) = 0$, find all values for x, y, and λ, such that

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$
$$g(x, y) = 0$$

The largest value of $f(x, y)$ is the maximum and the smallest value of $f(x, y)$ is the minimum.

This really amounts to solving the system of equations:

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g(x, y) &= 0 \end{aligned}$$

We can also maximize a function of three variables, $f(x, y, z)$, which results in a system of four equations and four unknowns: $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and the constraint $g(x, y, z) = 0$.

Example 1 Use Lagrange multipliers to find the extrema of $f(x, y) = 4x + 6y$ with the constraint $x^2 + y^2 = 13$.

A system with two constraints, g and h, and possibly three variables, will result in the system

$$\begin{aligned} &\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z) \\ &g(x, y, z) = 0 \\ &h(x, y, z) = 0 \end{aligned}$$

Example 2 Find the extrema of $f(x, y, z) = 3x - y - 3z$ subject to the constraints $x + y - z = 0$ and $x^2 + 2z^2 = 6$.

Example 3 The sum of three numbers, x, y, and z, is 20. Find the numbers that will minimize the value of $P = x^2 + 8x + y^2 + 2z$.