14.7 Maximum and Minimum Values

A function f has a local minimum value at (x_0, y_o) if $f(x_0, y_0) \le f(x, y)$ for all (x, y) near (x_0, y_0). (or a local max if $f(x_0, y_0) \ge f(x, y)$).

Theorem

If f has a local maximum or minimum value at (x_0, y_0) and the first partials of f exist then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. The point (x_0, y_0) is called a critical point of f.

Example 1 Find the critical points for the function: $f(x, y) = -x^3 - y^2 + x\,y + y - 1$. Which points give a local max, local min, or neither?

The Second Derivative Test (aka: The Determinant or Hessian Matrix)

Theorem

Suppose the second partials of the function $f(x, y)$ exist at (x_0, y_0) and that (x_0, y_0) is a critical point of f. Let

$$D = f_{xx}(x_0\, y_0)\, f_{yy}(x_0, y_0) - \left[f_{xy}(x_0, y_0)\right]^2$$

(a) if $D > 0$ and $f_{xx}(x_0, y_0) > 0$, (or $f_{yy}(x_0, y_0) > 0$) then $f(x_0, y_0)$ is a local minimum.

(b) if $D > 0$ and $f_{xx}(x_0, y_0) < 0$, (or $f_{yy}(x_0, y_0) < 0$) then $f(x_0, y_0)$ is a local maximum.

(c) if $D < 0$ then $f(x_0, y_0)$ is neither a local max or min (it is a saddle point.)

If $D = 0$ then $f(x_0, y_0)$ can be a local max, min, or neither. Additional analysis is needed to determine the nature of $f(x_0, y_0)$.

Note: the determinant can be written as $f_{xx}\, f_{yy} - (f_{xy})^2$ or as the determinant $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$. This is known as the Hessian matrix.

Example 2 Determine the nature of the critical points from Example 1.

Example 3 Find the local extrema and the saddle points for the function $f(x, y) = x\,y(1 - x - y)$.

Example 4 Find the points on the surface $x^2 y^2 z = 1$ that are closest to the origin.

Example 5 Find the absolute maximum and minimum value of the function $f(x, y) = 4x + 6y - 2x^2 - y^2$ where f is defined on the closed rectangular region $R = \{(x, y) \mid 0 \le x \le 4,\ 0 \le y \le 5\}$.