14.6 Directional Derivatives and the Gradient

Suppose we want to find the slope of the surface $z = f(x, y)$ not only parallel to the x-axis given by $f_x(x_0, y_0)$, or the y-axis given by $f_y(x_0, y_0)$, but rather in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$. This is called a **directional derivative**:

Theorem

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_{\mathbf{u}}(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

Example 1 Find the slope of the surface $f(x, y) = x^2 y - x y^3 + 4 x y$ in the direction $\mathbf{v} = \langle 3, 4 \rangle$ at the point (3, 1, 18).

The Gradient Vector

The directional derivative can be written in vector form:

$$\begin{aligned} D_{\boldsymbol{u}} f(x, y) &= f_x(x, y)\, a + f_y(x, y)\, b \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle a, b \rangle \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \boldsymbol{u} \end{aligned}$$

The first vector in the last expression is called the **gradient** of f and is denoted as

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

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Grad[x^2 y - x y^3 + 4 x y, {x, y}]
{4 y + 2 x y - y^3, 4 x + x^2 - 3 x y^2}
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Example 2 Show that the direction of the gradient of f is the direction that maximizes the rate of change in f.

💡 **Example 3** For $f(x, y,) = x^2 \cos(2y) + 2y$ (a) find the rate of change of f at the point (1, 0) in the direction of the point (2, −2). (b) In what direction does f have the maximum rate of change, and what is that maximum rate of change?

Example 4 The temperature in space at a point (x, y, z) is given by $T(x, y, z) = \frac{70}{1+2x^2+y^2+4z^2}$ where T is in Celsius and x, y, and z are in meters. In what direction does the temperature increase the fastest at the point (1, 2, −2), and what is the maximum rate of increase?

Example 5 For the surface given by $F(x, y, z) = k$, show that $\nabla F(x_0, y_0, z_0)$ is the *normal* vector to the surface at $P_0 = (x_0, y_0, z_0)$. Hint: let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ be any curve on the surface of F that passes through the point P_0.

💡 **Example 6** Find the equation of the tangent plane and the normal line to the surface $z^2 - 2x^2 - 2y^2 = 12$ at the point (1, −1, 4).