14.4 Tangent Planes and Linear Approximations

Tangent Planes

Recall from single-variable calculus a line tangent the function $y = f(x)$ at the point (x_0, y_0) is given by

$$y - y_0 = f'(x_0)(x - x_0)$$

(provided the function is differentiable at (x_0, y_0)), which can also be written

$$y = f(x_0) + f'(x_0)(x - x_0)$$

Similarly, for a function in two variables the analogy is a **tangent plane**. Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 1 Find the tangent plane to the function $f(x, y) = x^3 y - x^2 y^2 - 2y$ at the point $(2,1,2)$.

Example 2 Find the tangent plane to the function $f(x, y) = -x^2 - y^2 + 6x + 2y + xy - 6$ when $x = 3$ and $y = 2$. (Do you remember how to find the **normal** vector of the plane?) Make a plot of the surface, plane and normal vector.

Linear Approximations

The tangent plane provides us with a linearization approximation for a function f near the point (x_0, y_0). Using the equation of the tangent plane and solving for z we get

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 3 Linearize the function $f(x, y) = x^2 y + x y$ at the point (2, 3), and approximate $f(2.2, 2.9)$.

Theorem

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Example 4 Show that the function $f(x, y) = \sqrt{x + e^{4y}}$ is differentiable at (3, 0). Find a linearization for f and approximate $f(2.8, 0.1)$.

Differentials

Recall that the differential of $y = f(x)$ is given by $dy = f'(x)\,dx$, and represents the change in y using the linearization, as apposed to the actual change in y, or Δy (recall $\Delta x = dx$ and $\Delta y \approx dy$). For a function of several variables we have **the total differential**

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \approx \Delta z$$

Example 5 Find the total differential for $f(x, y) = x y^2 + 4 x y - x^2$. If x changes from 2 to 2.3 and y changes from 4 to 3.8, compare Δz and dz.