

14.3 Partial Derivatives

Recall the definition of a derivative:

$$\frac{d}{dx} f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

This means we are using two x values, $x_1 = x$ and $x_2 = x + h$, and as $h \rightarrow 0$ the two x values become arbitrarily close, resulting in the tangent slope or derivative. With multivariable functions we need to decide if we let the x 's "move" or the y 's "move". In doing so we treat the other variable as a constant, i.e.,

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} \quad \text{and} \quad f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

provided the limits exist. These are called **partial derivatives**.

Notation for Partial Derivatives

Let $z = f(x, y)$, then the notation for partial derivatives of f are:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = D_x f = \frac{\partial z}{\partial x} = z_x = f_1 \quad \text{and} \quad f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = D_y f = \frac{\partial z}{\partial y} = z_y = f_2$$

Note: when finding f_x treat y as a constant; when finding f_y treat x as a constant.

Example 1 Find f_x and f_y for $f(x, y) = 3xy^2 + \ln(xy)$.

Graphical Representation of Partial Derivatives

Example 2 Given $f(x, y) = x \cos(xy^2)$ find f_x and f_y at the point $(3, 0, 3)$

Example 3 Given $f(x, y, z) = xy^2e^{yz}$ find the first partials.

Example 4 Use implicit differentiation to find $\partial z / \partial x$ and $\partial z / \partial y$ for $yz = \ln(x + y)$

Higher Derivatives

A higher derivative is when a function is differentiated with respect to any variable more than once, i.e.

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}(x, y) = f_{yx}$$

which means take the derivative of f first with respect to y and then with respect to x .

Example 5 For the function $f(x, y) = y^3 + x^2 y - 5 x y^2$ find all the second partials: f_{xy} , f_{xx} , f_{yx} , f_{yy} . Are any the same?

Clairaut's Theorem

Suppose f is a real valued function and is continuous on an open set D in \mathbb{R}^2 , and suppose the partial f_{xy} and f_{yx} also exist and are continuous on D . Then $f_{xy} = f_{yx}$.

Example 6 For $f(x, y, z) = \cos(2x + yz)$ show that $f_{xz} = f_{zx}$.

Mathematica and Partial Derivatives.

Example 7 Given the partial differential equation: $u_{xx} + u_{yy} = 0$ (called Laplace's Equation), determine which of the following functions are solutions:

a) $u = x^2 + y^2$ b) $u = \ln\left(\sqrt{x^2 + y^2}\right)$ c) $u = x^2 - y^2$ d) $u = e^{-x} \cos(y) - e^{-y} \cos(x)$

Example 8 For $f(x, y) = x^2 + xy - 2y^3 - 13x + 2$ find where f_x and f_y are both simultaneously 0.

Example 9 For the surface $z = x^2 + 2xy^2 - 3y$ find the equation of the line on the plane $x = 2$ tangent to the surface z when $y = 1$.