14.2 Limit of a Function of Several Variables

Recall the definition of the limit of a function in one variable:

The $\lim_{x \to c} f(x) = L$ means that for every $\epsilon > 0$ (no matter how small) there is a corresponding $\delta > 0$ such that $|f(x) - L| < \epsilon$, provided that $0 < |x - c| < \delta$; that is

$$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$$

Using mathematical proof "symbology": $\forall\, \epsilon > 0 \;\; \exists \;\; \delta > 0 \;\ni\; \forall\, x, \;\; 0 < |x - c| < \delta \implies |f(x) - L| < \epsilon.$

ϵ - δ Definition of a Limit

Limit of a Function in Several Variables

Let f be a function with domain D, and let (a, b) be a point in D. Then, $\lim_{(x,y) \to (a,b)} f(x, y) = L$ means that for any $\epsilon > 0$ there is a corresponding $\delta > 0$ such that $|f(x, y) - L| < \epsilon$ whenever $\sqrt{(x-a)^2 + (y-b)^2} < \delta$.

Example 1 Find $\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ along the x-axis, and along the y-axis. Does the limit exist at (0, 0)? Find the limit along an arbitrary line through the origin.

Example 2 Find a probable limit for $f(x, y) = \frac{x - 2y}{x^2 - 2xy}$ as $(x, y) \to (4, 2)$. Why is this the actual value of the limit?

Example 3 Determine if the following limit exists: $\lim_{(x,y) \to (0,0)} \frac{xy + y^3}{x^2 + y^2}$.

Example 4 Determine if the following limit exists: $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$.

Continuity

A function is continuous at a point (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Example 5 Use polar coordinates to show that the function $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$ is not continuous at (0, 0).

Example 6 Suppose each of the following functions have the value 0 at (0, 0). Which of them are continuous at (0, 0) and which are discontinuous at (0, 0)?

a) $f(x, y) = \frac{x y}{\sqrt{x^2+y^2}}$

b) $f(x, y) = \frac{x - y}{\sqrt{x^2+y^2}}$

c) $f(x, y) = \frac{x^{7/3}}{x^2+y^2}$

d) $f(x, y) = x y \frac{x^2-y^2}{x^2+y^2}$

e) $f(x, y) = \frac{x^2 y^2}{x^2+y^4}$

f) $f(x, y) = \frac{x y^2}{x^2+y^4}$