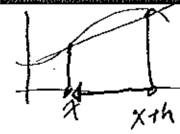
14.3 Partial Derivatives

Recall the definition of a derivative:

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$



This means we are using two x values, $x_1 = x$ and $x_2 = x + h$, and as $h \to 0$ the two x values become arbitrarily close, resulting in the tangent slope or derivative. With multivariable functions we need to decide if we let the x's "move" or the y's "move". In doing so we treat the other variable as a constant, i.e.,

$$f_x(x,\, y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} \quad \text{and} \quad f_y(x,\, y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

provided the limits exist. These are called **partial derivatives.**

Notation for Partial Derivatives

Let $z = f(x,\, y)$, then the notation for partial derivatives of f are:

$$f_x(x,\, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}f(x,\, y) = D_x f = \frac{\partial z}{\partial x} = z_x = f_1 \quad \text{and} \quad f_y(x,\, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}f(x,\, y) = D_y f = \frac{\partial z}{\partial y} = z_y = f_2$$

Note: when finding f_x treat y as a constant; when finding f_y treat x as a constant.

Example 1 Find f_x and f_y for $f(x,\, y) = 3xy^2 + \ln(xy)$.

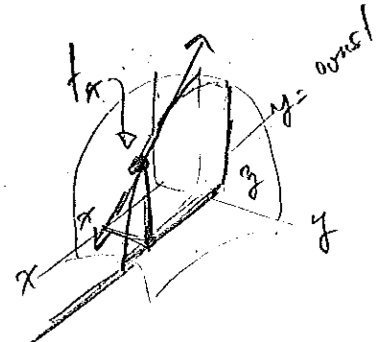
$f_x(x,y) = 3y^2 + \frac{1}{xy}\cdot y$

$= 3y^2 + \frac{1}{x}$

$f_y = 6xy + \frac{1}{xy}\cdot x$

$= 6xy + \frac{1}{y}$

Graphical Representation of Partial Derivatives



Example 2 Given $f(x,\, y) = x\cos(xy^2)$ find f_x and f_y at the point (3, 0, 3)

$f_x = \cos(xy^2) - xy^2\sin(xy^2)$

$f_x(3,0) = \cos(0) - 0$

$= 1$

$f_y = -2x^2y\sin(xy^2)$

$f_y(3,0) = 0$

Example 3 Given $f(x,\, y,\, z) = xy^2e^{yz}$ find the first partials.

$f_x = y^2e^{yz}$

$f_y = 2xye^{yz} + xy^2ze^{yz}$

$f_z = xy^3e^{yz}$

Example 4 Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$ for $yz = \ln(x+y)$

$\frac{\partial}{\partial x}(yz) = \frac{\partial}{\partial x}(\ln(x+y))$

$0\cdot z + y\frac{\partial z}{\partial x} = \frac{1}{x+y}(1+0)$

$y\frac{\partial z}{\partial x} = \frac{1}{x+y}$

$\frac{\partial z}{\partial x} = \frac{1}{y(x+y)}$

$\frac{\partial}{\partial y}(yz) = \frac{\partial}{\partial y}(\ln(x+y))$

$1\cdot z + \frac{\partial z}{\partial y}y = \frac{1}{x+y}\cdot(0+1)$

$z + \frac{\partial z}{\partial y}y = \frac{1}{x+y}$

$\frac{1}{y}\frac{\partial z}{\partial y}y = \left(\frac{1}{x+y} - z\right)\frac{1}{y}$

$\frac{\partial z}{\partial y} = \frac{1}{y(x+y)} - \frac{z}{y}$

Higher Derivatives

A higher derivative is when a function is differentiated with respect to any variable more than once, i.e.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\,\partial y}=f_{yx}(x,y)=f_{yx}$$

which means take the derivative of f first with respect to y and then with respect to x.

Example 5 For the function $f(x, y)=y^3+x^2y-5xy^2$ find all the second partials: f_{xy}, f_{xx}, f_{yx}, f_{yy}. Are any the same?

$f_x = 2xy-5y^2$

$f_{xx}=2y$

$f_{xy}=2x-10y$

$f_{yx}=2x-10y$

same

$f_y=3y^2+x^2-10xy$

$f_{yy}=6y-10x$

Clairaut's Theorem

Suppose f is a real valued function and is continuous on an open set D *in* $\mathbb{R}^2$, and suppose the partial f_{xy} and f_{yx} also exist and are continuous on D. Then $f_{xy}=f_{yx}$.

Example 6 For $f(x, y, z)=\cos(2x+yz)$ show that $f_{xxz}=f_{xzx}=f_{zxx}$.

$f_x = -2\sin(2x+yz)$

$f_{xx}=-4\cos(2x+yz)$

$f_{xxz}=4y\sin(2x+yz)$

$f_{xz}=-2y\cos(2x+yz)$

$f_{xzx}=4y\sin(2x+yz)$

$f_z=-y\sin(2x+yz)$

$f_{zx}=-2y\cos(2x+yz)$

$f_{zxx}=4y\sin(2x+yz)$

Mathematica and Partial Derivatives.

Example 7 Given the partial differential equation: $u_{xx}+u_{yy}=0$ (called Laplace's Equation), determine which of the following functions are solutions:

a) $u=x^2+y^2$

$u_x=2x$ $u_{xx}=2$

$u_y=2y$ $u_{yy}=2$

$2+2\neq 0$

No

b) $u=\ln\left(\sqrt{x^2+y^2}\right)$

see attached

c) $u=x^2-y^2$

$u_x=2x$ $u_{xx}=2$

$u_y=-2y$ $u_{yy}=-2$

$2+(-2)=0$ yes.

d) $u=e^{-x}\cos(y)-e^{-y}\cos(x)$

see attached.

Example 8 For $f(x, y)=x^2y-xy^2-y$ find where f_x and f_y are both simultaneously 0.

$f_x=2xy-y^2$

$0=2xy-y^2$

$0=y(2x-y)$

$y=0$ or $y=2x$

$f_y=x^2-2xy-1$

$0=x^2-2xy-1$

$y=0$

$\Rightarrow 0=x^2-1$

$0=(x-1)(x+1)$

$x=1, x=-1$

$(1,0)$ & $(-1,0)$

$y=2x$

$\Rightarrow 0=x^2-4x^2-1$

$0=-3x^2-1$

$3x^2=-1$

$\emptyset$

Example 9 For the surface $z=x^2+2xy^2-3y$ find the equation of the line on the plane $x=2$ tangent to the surface z when $y=1$.

The point is $z=2^2+2(2)(1)^2-3(1)$

$=4+4-3$

$=5$

$P(2,1,5)$

$z_y=4xy-3$

$z_y(2,1)=5$

slope

$z-5=5(y-1)$

$z=5y-5+5$

$z=5y$

if $y=t$

line $=\begin{cases}x=2\\ y=t\\ z=5t\end{cases}$ or $r(t)=\langle 2,1,5\rangle+\langle 0,1,5\rangle t$

Ex 7

b) $u = \ln\left(\sqrt{x^2+y^2}\right)$

$$u = \frac{1}{2}\ln(x^2+y^2)$$

$$u_x = \frac{1}{\cancel{2}(x^2+y^2)} \cdot \cancel{2}x$$

$$u_x = \frac{x}{x^2+y^2}$$

$$u_{xx} = \frac{1(x^2+y^2) - 2x \cdot x}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$u_{xx} = \frac{y^2-x^2}{(x^2+y^2)^2} \Rightarrow u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2} \Rightarrow u_{xx} = -u_{yy}$$

$$u_{xx} + u_{yy} = 0$$

✓ yes.

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d) $u = e^{-x}\cos(y) - e^{-y}\cos(x)$

$u_x = -e^{-x}\cos(y) + e^{-y}\sin(x)$ $\qquad$ $u_y = -e^{-x}\sin(y) + e^{-y}\cos(x)$

$u_{xx} = e^{-x}\cos(y) + e^{-y}\cos(x)$ $\qquad$ $u_{yy} = -e^{-x}\cos(y) - e^{-y}\cos(x)$

$$u_{xx} + u_{yy} = e^{-x}\cancel{\cos(y)} + e^{-y}\cancel{\cos(x)} - e^{-x}\cancel{\cos(y)} - e^{-y}\cancel{\cos(x)}$$

$$= 0$$

yes ✓