14.2 Limit of a Function of Several Variables

Recall the definition of the limit of a function in one variable:

The $\lim_{x\to c} f(x)=L$ means that for every $\epsilon>0$ (no matter how small) there is a corresponding $\delta>0$ such that $|f(x)-L| < \epsilon$, provided that $0<|x-c|<\delta$; that is

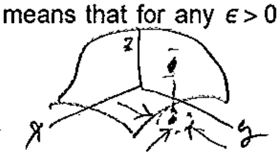
$$0<|x-c|<\delta \implies |f(x)-L|<\epsilon$$

Using mathematical proof "symbology": $\forall\, \epsilon>0\ \exists\ \delta>0 \ni \forall x,\ 0<|x-c|<\delta \implies |f(x)-L|<\epsilon$.

ϵ - δ Definition of a Limit For all epsilon greater than 0, there exists a delta greater than 0 such that for all x, $0<|x-c|<\delta$ implies $|f(x)-L|<\varepsilon$.

Limit of a Function in Several Variables

Let f be a function with domain D, and let (a, b) be a point in D. Then, $\lim_{(x,y)\to(a,b)} f(x,y)=L$ means that for any $\epsilon>0$ there is a corresponding $\delta>0$ such that $|f(x,y)-L|<\epsilon$ whenever $\sqrt{(x-a)^2+(y-b)^2}<\delta$.



Example 1 Find $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ along the x-axis, and along the y-axis. Does the limit exist at (0, 0)? Find the limit along an arbitrary line through the origin.

x-axis $\Rightarrow y=0$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{(x,0)\to(0,0)} \frac{x^2-0^2}{x^2+0^2} = \lim_{x\to 0} 1 = 1$$

y-axis $\Rightarrow x=0$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{0^2-y^2}{0^2+y^2} = \lim_{y\to 0} -1 = -1$$

$y=mx$

$$L = \lim_{(x,mx)\to(0,0)} \frac{x^2-(mx)^2}{x^2+(mx)^2} = \lim_{x\to 0} \frac{\cancel{x^2}(1-m^2)}{\cancel{x^2}(1+m^2)} = \frac{1-m^2}{1+m^2}$$

Example 2 Find a probable limit for $f(x,y)=\frac{x-2y}{x^2-2xy}$ as $(x,y)\to(4,2)$. Why is this the actual value of the limit?

$$\lim_{(x,y)\to(4,2)} \frac{x-2y}{x^2-2xy} = \lim_{(x,y)\to(4,2)} \frac{\cancel{x-2y}}{x(\cancel{x-2y})} = \lim_{(x,y)\to(4,2)} \frac{1}{x} = \frac{1}{4}$$

Example 3 Determine if the following limit exists: $\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2}$.

x-axis $\Rightarrow y=0$

$$L = \lim_{(x,0)\to(0,0)} \frac{0+0}{x^2+y^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$$

along $y=x$

$$L = \lim_{(x,x)} \frac{x^2+x^3}{x^2+x^2} = \lim_{x\to 0} \frac{x^2+x^3}{2x^2} = \lim_{x\to 0} \frac{1+x}{2} = \frac{1}{2}$$

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Example 4 Determine if the following limit exists: $\lim\limits_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$.

along x-axis $\Rightarrow y=0$

$\lim\limits_{(x,0)\to(0,0)} \frac{0}{x^4+y^2} = 0$

$L=0$ along y-axis

along $y=kx$

$\lim\limits_{(x,kx)\to(0,0)} \frac{x^2 kx}{x^4+kx^2}$

$= \lim\limits_{x\to 0} \frac{x^3 k}{x^4+kx^2}$

$= \lim\limits_{x\to 0} \frac{xk}{x^2+k}$

$= \frac{0}{k}$

$= 0$

along $y=x^2$

$\lim\limits_{(x,x^2)\to(0,0)} \frac{x^2 x^2}{x^4+(x^2)^2}$

$= \lim\limits_{x\to 0} \frac{x^4}{2x^4}$

$= \frac{1}{2}$ //

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Continuity

A function is continuous at a point (a, b) if $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Example 5 Use polar coordinates to show that the function $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$ is not continuous at (0, 0).

Polar coordinates

$x = r\cos(\theta)$

$y = r\sin(\theta)$

as $r\to 0$ $(x,y)\to(0,0)$

$\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim\limits_{r\to 0} \frac{r^2\cos^2(\theta) - r^2\sin^2(\theta)}{r^2\cos^2\theta + r^2\sin^2(\theta)}$

$= \lim\limits_{r\to 0} \frac{\cos^2(\theta)-\sin^2(\theta)}{1}$

$L = \cos(2\theta)$

The limit depends on the angle of approach to $(0,0)$.

since $L \neq 0$ always f is not cont.

Example 6 Suppose each of the following functions have the value 0 at (0, 0). Which of them are continuous at (0, 0) and which are discontinuous at (0, 0)?

a) $f(x, y) = \frac{xy}{\sqrt{x^2+y^2}}$ b) $f(x, y) = \frac{x-y}{\sqrt{x^2+y^2}}$ c) $f(x, y) = \frac{x^{7/3}}{x^2+y^2}$

d) $f(x, y) = xy\frac{x^2-y^2}{x^2+y^2}$ e) $f(x, y) = \frac{x^2y^2}{x^2+y^4}$ f) $f(x, y) = \frac{xy^2}{x^2+y^4}$

see below

Ex6 a) use polar

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r\cos(\theta)\, r\sin(\theta)}{\sqrt{r^2}}$$

$$= \lim_{r\to 0} \frac{r^2\cos(\theta)\sin(\theta)}{|r|}$$

$$= \lim_{r\to 0} |r| \cos(\theta)\sin(\theta)$$

$$= 0$$ continuous since $f(0,0)=0$.

b) along x-axis.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,0)\to(0,0)} \frac{x-0}{\sqrt{x^2+y^2}}$$

$$= \lim_{x\to 0} \frac{x}{|x|}$$

$$\Rightarrow \lim_{x\to 0^-} \frac{x}{|x|} = -1$$ not cont.

$$\lim_{x\to 0^+} \frac{x}{|x|} = +1$$

c) $f(x,y) = \dfrac{x^{7/3}}{x^2+y^2}$ use the squeeze Thm. twice.

if $x>0$ then $0 < \dfrac{x^{7/3}}{x^2+y^2} < \dfrac{x^{7/3}}{x^2}$

and $0 < \dfrac{x^{7/3}}{x^2+y^2} < x^{1/3}$

$$\lim_{(x,y)\to(0,0)} 0 < \lim_{(x,y)\to(0,0)} \frac{x^{7/3}}{x^2+y^2} < \lim_{x\to 0} x^{1/3}$$

$$0 < \lim_{(x,y)\to(0,0)} \frac{x^{7/3}}{x^2+y^2} < 0$$

$\Rightarrow L=0$

if $x<0$

$$\frac{x^{7/3}}{x^2} < \frac{x^{7/3}}{x^2+y^2} < 0$$

similarly $\lim_{(x,y)\to(0,0)} \dfrac{x^{7/3}}{x^2+y^2} = 0$

if $x=0$ $\lim_{(0,y)\to(0,0)} = \dfrac{0^{7/3}}{0^2+y^2} = 0$

$L=0$ cont.

d) $f(x,y) = xy\,\dfrac{x^2-y^2}{x^2+y^2}$ use polar

$$L = \lim_{r\to 0} r\cos(\theta)\, r\sin(\theta)\,\frac{r^2\cos^2(\theta) - r^2\sin^2(\theta)}{r^2(\cos^2\theta) + r^2\sin^2(\theta)}$$

$$= \lim_{r\to 0} r^2\cos(\theta)\sin(\theta)\underbrace{\left(\cos^2(\theta) - \sin^2(\theta)\right)}_{1}$$

$$= \underline{\underline{0}}$$

Cont.

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e) Use the ε-δ def. to check if $L = 0$.
show $|f(x) - 0| < \varepsilon$ whenever $\sqrt{x^2+y^2} < \delta$

$f(x,y) = \dfrac{x^2y^2}{x^2+y^4}$ notice that $y^2 < x^2+y^2$ and $\dfrac{x^2}{x^2+y^4} < 1$
for all $(x,y) \neq (0,0)$

$\Rightarrow \sqrt{y^2} < \sqrt{x^2+y^2}$; let $\sqrt{y^2} < \sqrt{x^2+y^2} < \delta$ or $y^2 < \delta^2$

then $|f(x,y) - 0| = \left|\dfrac{x^2y^2}{x^2+y^4}\right|$

$= \left|\dfrac{x^2}{x^2+y^4}\right| y^2$

$< y^2$ choose $\varepsilon = \delta^2$

$< \delta^2$ so $\sqrt{\varepsilon} = \delta$

$< \varepsilon$ and since $\sqrt{y^2} < \delta$

$\therefore |f(x,y) - 0| < \varepsilon$ whenever $\sqrt{x^2+y^2} < \delta$ for $\underline{\delta = \sqrt{\varepsilon}}$.

Cont.

f) $f(x,y) = \dfrac{xy^2}{x^2+y^4}$ along x-axis or y-axis $L=0$

along $x=y^2$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{(y^2,y)\to(0,0)} \frac{y^2y^2}{(y^2)^2+y^4}$

$\displaystyle = \lim_{y\to 0} \frac{y^4}{2y^4}$

$\displaystyle = \lim_{y\to 0} \frac{1}{2}$

$= \dfrac{1}{2}$

Not cont.