7.5 Solving Homogeneous Linear System with Constant Coefficients

For the differential equation $\mathbf{x}' = A\,\mathbf{x}$, suppose A has distinct real eigenvalues λ_1 and λ_2, and associated eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. The solution to the differential equation is

$$\mathbf{x} = c_1\,\mathbf{v}_1\,e^{\lambda_1 t} + c_2\,\mathbf{v}_2\,e^{\lambda_2 t}$$

where the constants c_1 and c_2 are found using initial conditions.

Example 1 Solve the system of differential equations: $\begin{aligned} x'_1 &= x_1 + 2\,x_2 \\ x'_2 &= 3\,x_1 + 2\,x_2 \end{aligned}$ or in matrix form $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\mathbf{x}$.

Linearly Independent Solutions

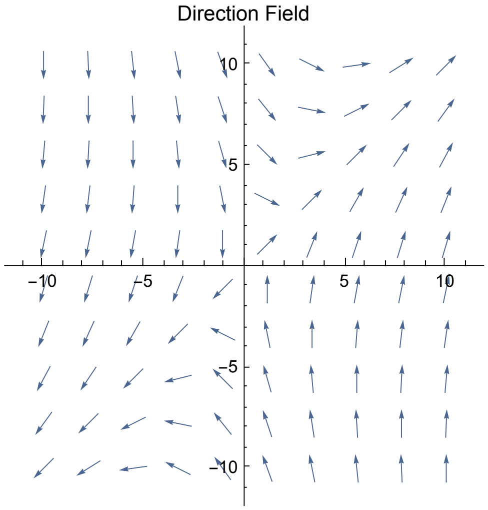
To determine if the solutions $\mathbf{x}_1 = \mathbf{v}_1\,e^{\lambda_1 t}$ and $\mathbf{x}_2 = \mathbf{v}_2\,e^{\lambda_2 t}$ are linearly independent evaluate their Wronskian: $W = |\,\mathbf{x}_1 \quad \mathbf{x}_2\,|$. If $W \neq 0$ the solutions are independent and form a fundamental set for the general solution.

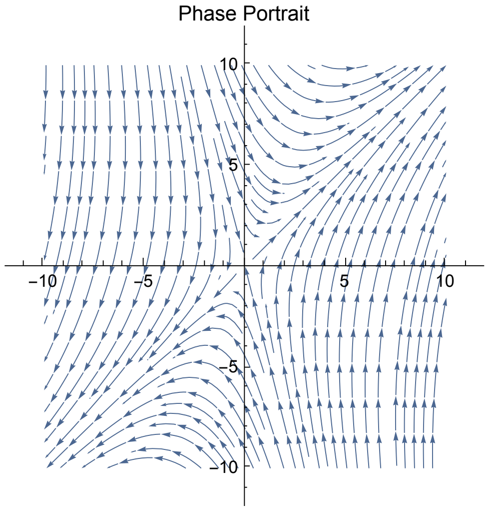
Example 2 Find the Wronskian for example 1.

Example 3 Solve the IVP $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x}$; $\mathbf{x}(0) = \begin{pmatrix} -1 \\ 9 \end{pmatrix}$.

Direction Fields and Phase Portraits

The direction field for example 3 is below. Plot several trajectories for the system.





Example 4 Plot and analyze the eigenvectors and associated eigenvalues on the graphs above.

Example 5 Solve the IVP: $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}\mathbf{x}$; $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Example 6 Find the general solution to $\mathbf{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}\mathbf{x}$