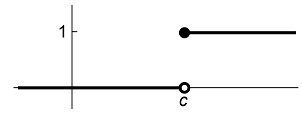
6.3 Step Functions

A step function is a function that has one or more jump discontinuities, e.g. the greatest integer function. A useful step function to tackle discontinuous forcing functions is the **Heaviside** function or **unit step** function, defined as

$$u(t) = H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \quad \text{or} \quad u_c(t) = H(t - c) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases} \implies$$



Example 1 Make a graph of: $f(t) = u_2(t) - u_5(t)$

Example 2 Find $\mathcal{L}\{u_c(t)\}$.

Example 3 Sketch and rewrite the function using the unit step function: $f(t) = \begin{cases} 0 & t < 2 \\ \cos(6(t-2)) + 2 & 2 \leq t < 4 \\ 0 & t \geq 4 \end{cases}$

Theorem: Laplace of a Discontinuous Function

$$\mathcal{L}\{u_c(t)\, f(t-c)\} = e^{-cs} F(s) \qquad s > a$$

Proof:

Example 4 Find the Laplace transform for: $f(t) = \begin{cases} 0 & t < 1 \\ t^2 - 2t + 2 & t \geq 1 \end{cases}$

Example 5 Find $\mathcal{L}\{u_3(t)\cos(4t-12)\}$

Example 6 Find the Laplace transform for: $f(t) = u_2(t)\, t^2$.

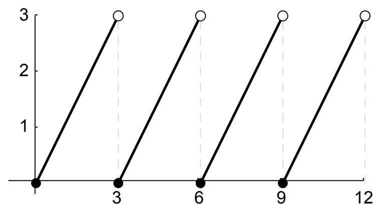
Example 7 Find the Inverse Laplace transform for: $Y(s) = \frac{e^{-2s}}{s^2+s-2}$.

Example 8 Find the inverse Laplace transform for $F(s) = \frac{e^{-3s}}{s(s^2+4)}$

Challenge Find $\mathcal{L}\{u_4(t)\sin(t)\}$ Hint: use the sine sum formula to create $\sin(t)$.

Periodic Functions

A periodic function is a function f that satisfies $f(t+T) = f(t)$ for all $t \geq 0$ for some fixed number T. The period of the function is T. *Example:*



A single period of the function is called a *window*, and the window of the function is denoted: $f_T(t) = \begin{cases} f(t) & 0 \leq t < T \\ 0 & \text{elsewhere} \end{cases}$

Theorem: The Laplace Transform of a Periodic Function

Let $f(t)$ be a periodic function with period T on $0 \leq t < \infty$. Then,

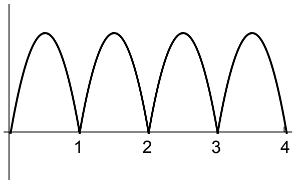
$$\mathcal{L}\{f(t)\} = \frac{\mathcal{L}\{f_T(t)\}}{1-e^{-Ts}}$$

$$= \frac{\int_0^T e^{-st} f(t)\, dt}{1-e^{-Ts}}$$

Proof:

Example 9 Find the Laplace transform of $f(t)$ given above.

Example 10 Find the Laplace transform for: $f_T(t) = \begin{cases} t - t^2 & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$.



Example 11 Graph the window and find the Laplace transform for $f_T(t) = \begin{cases} 2t & 0 < t \le 1/2 \\ -2t + 2 & 1/2 < t \le 3/2 \\ 2t - 4 & 3/2 < t \le 2 \end{cases}$.