6.2 Solutions to Initial Value Problems Using Laplace Transforms

Solving an IVP using Laplace transforms results in transforming an IVP problem to a purely algebraic equation (non-integral calculus) and then converting back to the DE solution.

Theorem: Laplace Transform of the Derivative

Suppose $y = f(t)$ is continuous and $y' = f'(t)$ is piecewise continuous on an interval I, then

$$\mathcal{L}\{y'(t)\} = s\,Y(s) - y(0)$$

Corollary

Suppose the functions $f,\ f',\ f'',\ \ldots f^{(n-1)}$ are continuous, and $f^{(n)}$ is piecewise continuous on an interval I, then

$$\mathcal{L}\left\{y^{(n)}(t)\right\} = s^n\,Y(s) - s^{n-1}\,y(0) - s^{n-2}\,y'(0) - \cdots - s\,y^{(n-2)}(0) - y^{(n-1)}(0)$$

Example 1 Find $\mathcal{L}\{y'''\}$ if $y(0) = 2$, $y'(0) = -1$, and $y''(0) = 7$.

Inverse Laplace Transform

The inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$ converts a function of s back into a function of t. This process is usually done using a table of Laplace transforms.

Example 2 Find the inverse Laplace transform for $F(s) = \frac{12}{s^4} - \frac{5}{s+2} + \frac{s+6}{s^2+4}$

Partial Fraction Decomposition

Solving differential equations involves a bit of partial fraction decomposition and inverse transforms.

Example 3 Find the partial fraction decomposition for $F(s) = \frac{6s+2}{s^2+4s+3}$, and the corresponding inverse Laplace transform.

Mathematica Commands

```
InverseLaplaceTransform[(6 s + 2)/(s^2 + 4 s + 3), s, t] // Expand
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$8\,e^{-3t} - 2\,e^{-t}$

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Example 4 Find the inverse Laplace transform for: $Y(s) = \frac{s^2+10s+33}{(s+1)(s^2+6s+13)}$.

Solving an IVP Using Laplace Transforms

Example 5 Use Laplace transforms to solve the IVP: $y'' + 3y' + 2y = 4t$, $y(0) = 3$, $y'(0) = 1$.

Example 6 Use Laplace transforms to solve the IVP: $y'' + 4y = 6\cos(t)$, $y(0) = 12$, $y'(0) = 4$.

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