6.1 Laplace Transforms

In mathematics, **transform theory** is the study of transforms. The essence of transform theory is that by a suitable choice of basis for a vector space a problem may be simplified, e.g., calculating the volume of a cone in rectangular coordinates is simplified by converting to cylindrical coordinates.

In calculus, an integral transform involves taking the integral of function, e.g.

$$\int_{\alpha}^{\beta} K(s,\, t)\, f(t)\, dt$$

The function $K(s, t)$ is called the **kernel** of the transform. An integral transform maps a function or equation that may be unwieldy in its original domain into another domain. Once the equation is solved in the "simpler" domain, it is then untransformed (using an inverse transform) back into its original domain.

The Laplace Transform

The Laplace transform, denoted as $\mathcal{L}\{f(t)\}$ or $F(s)$ uses the kernel $K(s, t) = e^{-st}$, and is given by

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0}^{\infty} e^{-st} f(t)\, dt$$

provided the integral exists. If the integral does not exists, then $f(t)$ does not have a Laplace transform.

Example 1 Find the Laplace transform for $f(t) = 1$.

Example 2 Find the Laplace transform for $f(t) = t$.

In general, $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, example: $\int_{0}^{\infty} t^4\, e^{-st}\, dt = -\frac{e^{-st}\left(24+24\,s\,t+12\,s^2\,t^2+4\,s^3\,t^3+s^4\,t^4\right)}{s^5}\Big|_{0}^{\infty} =$

Example 3 Find the Laplace transform for $f(t) = e^{rt}$.

Example 4 Find the Laplace transform for $f(t) = \cos(a\,t)$.

Example 5 Show that the Laplace transform is a linear operator, i.e. $\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1 F[s] + c_2 G[s]$.

Example 6 Find the Laplace transform for $f(t) = 3t^2 + 2e^{-t} - 5\sin(3t)$

Mathematica Commands

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LaplaceTransform[3 t^2 + 2 e^-t - 5 Sin[3 t], t, s]
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$$\frac{6}{s^3} + \frac{2}{1+s} - \frac{15}{9+s^2}$$