3.9 Forced Vibrations with Damping

Consider the mechanical vibration formula with a forcing function in the form $F(t) = F_0 \cos(\omega t)$,

$$m u'' + \gamma u' + k u = F_0 \cos(\omega t)$$

The general solution will be the complimentary solution to the homogeneous equation plus the particular solution:

$$u(t) = C_1 u_1(t) + C_2 u_2(t) + A \cos(\omega t) + B \sin(\omega t) = u_c(t) + U(t)$$

The complimentary solution depends on the roots of the homogeneous equation $m u'' + \gamma u' + k u = 0$, and are always either both negative and real, or complex conjugates with a negative real part. These two cases result in the complimentary solutions

1. $u_c(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$, or
2. $u_c(t) = e^{r t}(A \cos(\omega_0 t) + B \sin(\omega_0 t))$

💡 In both cases $u_c(t) \to 0$ as $t \to \infty$, and are called **transient solutions**. The remaining terms from $U(t) = A \cos(\omega t) + B \sin(\omega t)$ do not die out and is called the **steady-state solution** or the **forced response**.

Example 1 Approximate the transient solution and steady-state solution for the equation: $4 u'' + 4 u' + 17 u = 20 \cos(6 t)$, $u_o = 2$ and $u_0 = 0$. Rewrite the steady-state in the form $U(t) = R \cos(\omega t - \delta)$. (Helpful hint: $C_1 = \frac{7190}{3341}$, $C_2 = \frac{3019}{6682}$, $A = \frac{-508}{3341}$, $B = \frac{96}{3341}$)

Undamped Forced Vibrations

An undamped forced vibration equation has the form

$$m u'' + k u = F_0 \cos(\omega t)$$

Two interesting cases result (a) when $\omega_0 \neq \omega$ and (b) $\omega_0 = \omega$.

Case I: $\omega_0 \neq \omega$ Beats

The general solution to the undamped forced equation is

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0{}^2 - \omega^2)} \cos(\omega t)$$

If $u(0) = 0$ and $u'(0) = 0$, then the resulting motion is powered by the forcing function alone, and the constants become

$$C_1 = \frac{-F_0}{m(\omega_0{}^2 - \omega^2)} \text{ and } C_2 = 0$$

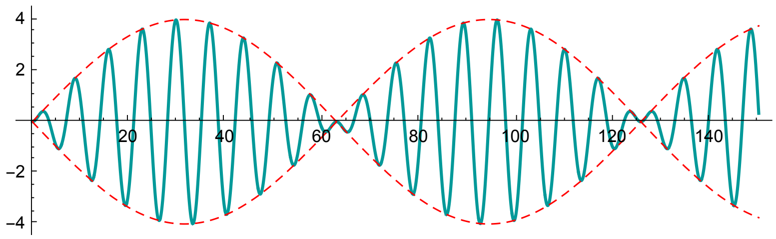
This gives the solution

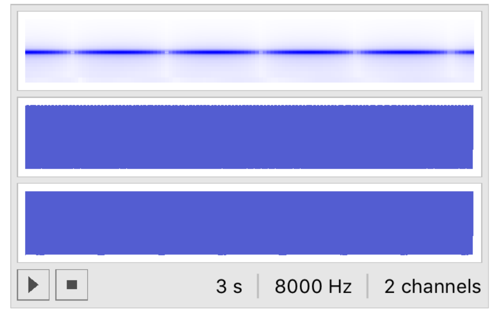
$$u(t) = \frac{F_0}{m(\omega_0{}^2 - \omega^2)} (\cos(\omega t) - \cos(\omega_0 t))$$

$$\begin{aligned} u(t) &= \frac{F_0}{m(\omega_0{}^2 - \omega^2)} (\cos(\omega t) - \cos(\omega_0 t)) \\ &= \frac{2 F_0}{m(\omega_0{}^2 - \omega^2)} \sin\left(\frac{\omega_0 - \omega}{2} t\right) \sin\left(\frac{\omega_0 + \omega}{2} t\right) \end{aligned}$$

If $\omega_0 \approx \omega$, then $|\omega_0 - \omega|$ is small compared to $\omega_0 + \omega$. This results in a rapidly oscillating function with a slowly oscillating amplitude function.

Example: $u(t) = 4\sin(0.05\,t)\sin(0.9\,t)$ (The sound example *Mathematica* code is `Play[{Sin[2000 t],Sin[2010 t]},{t,0,3}]`.





Case II: $\omega_0 = \omega$ Resonance

When $\omega_0 = \omega$ the solution to the undamped forced equation becomes

$$u(t) = C_1\cos(\omega_0\,t) + C_2\sin(\omega_0\,t) + \frac{F_0}{2\,m\,\omega_0}\,t\sin(\omega_0\,t)$$

The term $\frac{F_0}{2\,m\,\omega_0}\,t\sin(\omega_0\,t)$ causes the solution to become unbounded as $t \to \infty$. This phenomenon is called **resonance**.

Example 2 Solve the IVP $u'' + 16\,u = 10\cos(4\,t)$ $u(0) = 0$ and $u'(0) = 0$.

Example 3 For a damped-forced oscillation in the form $u'' + \gamma u' + k\,u = F\cos(\omega\,t)$, define the **gain** as $G(\omega) = \frac{1}{\sqrt{(k-\omega^2)^2 + \gamma^2\,\omega^2}}$. Consider the equation $u'' + 0.2\,u' + 49\,u = \cos(\omega\,t)$, $u(0) = 0$ and $u'(0) = 0$. Graph $G(\omega)$ to find the driving frequency ω that produces resonance.