3.8 Mechanical Vibrations

We want to study the harmonic motion of a mass on a spring by considering all the forces acting on the mass. These ideas can be expanded to many mechanical and electrical vibrations. In this case, there are four forces acting on a mass to consider:

1. The force of gravity acting downward (in the positive direction) on the mass: F_g
2. A damping force which is in the opposite direction of movement: F_d
3. The force from the spring in returning the mass to it's equilibrium position: F_s
4. An external force acting on the mass: F_e

We also know that force = mass × acceleration. If we let $u(t)$ be the displacement of the mass at any time t, we have $m\,u'' = f(t)$ where $f(t)$ is the total net force acting on the mass at anytime t. We'll also need to specify the displacement of the spring due to the mass for Hooke's Law.

Derivation of the Mechanical Vibration Formula: $m\,u'' + \gamma\,u' + k\,u = F(t)$

Example 1 A mass weighing 8 pounds stretches a spring 6 inches. Suppose the mass is stretched (in the positive direction) an additional 18 inches and released. The mass is in a medium that exerts a damping resistance of 6 lbs when the mass has a velocity of 2 ft/sec. Formulate an initial value problem for the position of the mass at anytime t.

Undamped Free-Vibrations

An undamped free vibration is where there is no external force nor a damping force. The equation for the displacement is therefore

$$m\,u'' + k\,u = 0$$

The general solution has the form $u(t) = A\cos(\omega_0\, t) + B\sin(\omega_0\, t)$, where $\omega_0 = \sqrt{k/m}$.

Example 2 A mass of 2 kg stretches a spring 0.2 meters beyond its natural length. The mass is pulled down an additional 0.5 m and released with a velocity of 3 meters per second. Find the position of the mass at any time *t*.

Example 3 Rewrite the solution from *Example 2* in the form $u(t) = R\cos(\omega_0\, t - \delta)$, where $R = \sqrt{A^2 + B^2}$ and $\tan(\delta) = B/A$. (Also, $A = R\cos(\delta)$ and $B = R\sin(\delta)$).

Damped Free-Vibrations

Example 4 Solve the general **damped free-vibration** equation: $m\,u'' + \gamma\,u' + k\,u = 0$, and find the characteristics of all possible solutions.

Types of Damping

For the equation $m\,u'' + \gamma\,u' + k\,u = 0$, if

1. If $\gamma > 2\sqrt{m\,k}$ the vibration is **overdamped**. Solution: $u(t) = C_1\,e^{-r_1 t} + C_2\,e^{-r_2 t}$ (no zeros)
2. If $\gamma = 2\sqrt{m\,k}$ the vibration is **critically damped**. Solution $u(t) = C_1\,e^{-r\,t} + C_2\,t\,e^{-r\,t}$
4. If $\gamma < 2\sqrt{m\,k}$ the vibration is **underdamped**. Solution: $u(t) = e^{\frac{-\gamma}{2m}t}(A\cos(\mu\,t) + B\sin(\mu\,t))$

Example 5 Determine the damping for the following:

a) $0.5\,u'' + 6\,u' + 18\,u = 0$
b) $2\,u'' + 12\,u' + 3\,u = 0$
c) Equation from *Example 1*

Example 6 Consider the equation $u'' + \beta\,u' + u = 0$, with $u(0) = 1$ and $u'(0) = 0$. Determine the damping for $\beta = 1,\ 2,$ and 3. Which value β results in the greatest decay rate?

Electrical Circuits

Simple electrical circuits involve current, I (in amperes), the resistance R (in ohms), the capacitance C (in farads), and the inductance L (in henrys). The current is a function of time t. Another important quantity is the total charge Q (in coulombs) on the capacitor at time t. The relationship between a charge Q and current I is

$$I = dQ/dt$$

Kirchhoff's second law of current flow states: *In a closed circuit, the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit.*

Also, using the elementary laws of electricity:

The voltage drop across the resistor is $I\,R$.
The voltage drop across the capacitor is Q/C.
The voltage drop across the inductor is $L\,\frac{dI}{dt}$.

Using Kirchhoff's law we get:

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t)$$

Substituting $I = dQ/dt$ we have the differential equation:

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

The initial conditions are $I(t_0) = I_0$ and $I'(t_0) = I'_0$

Example 7 A series circuit has a capacitor of 10^{-5} farad, a resistor of 3×10^2 ohms, and an inductor of 0.2 henry. The initial charge on the capacitor is 10^{-6} coulomb and there is no initial current. Find the charge Q on the capacitor at any time t.

Example 8 If a series circuit has a capacitor of $C = 0.8\times 10^{-6}$ farad and an inductor of $L = 0.2$ henry, find the resistance R so that the circuit is critically damped.