3.7 Variation of Parameter

Consider the differential equation $P(t)\,y'' + Q(t)\,y' + R(t) = G(t)$. We've looked at various methods to find the solution:

Method of Reduction of Order
Pros: Straight forward in concept.
Cons: not always easy to implement. Must have one complimentary solution to start with.

Method of Undetermined Coefficients
Pros: Do not need to know the complimentary solution to find the particular solution.
Cons: Only applicable for special cases of $g(t)$ (i.e. polynomial, exponential, sine or cosine)

Method of Variation of Parameter
Pros: can be used when the coefficients are variable (i.e., functions, not just constants)
Cons: the complimentary solutions must be known. Final solution may have to be numerically determined.

Variation of Parameter

If the equation $P(t)\,y'' + Q(t)\,y' + R(t) = G(t)$ is divided through by $P(t)$, then the differential equation can be written as

$$y'' + p(t)\,y' + q(t)\,y = g(t) \tag{1}$$

and we can assume a solution exists on the interval where p, q, and g are continuous containing t_0.

Let y_1 and y_2 be solutions to the corresponding homogeneous equation, (e.g., $y_h = C_1 y_1 + C_2 y_2$). Suppose a particular solution is in the form $y_p = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$ for some unknown functions u_1 and u_2 (we are "varying" the parameters C_1 and C_2 to be functions).

To find u_1 and u_2, first find $\dot{y}_p$

$$\dot{y}_p = \dot{u}_1 y_1 + u_1 \dot{y}_1 + \dot{u}_2 y_2 + u_2 \dot{y}_2$$

To simplify the algebra later, assume (2) $\dot{u}_1 y_1 + \dot{u}_2 y_2 = 0$. (More about this assumption later.) This means

$$\dot{y}_p = u_1 \dot{y}_1 + u_2 \dot{y}_2$$

and gives us $\ddot{y}_p$,

$$\ddot{y}_p = \dot{u}_1 \dot{y}_1 + u_1 \ddot{y}_1 + \dot{u}_2 \dot{y}_2 + u_2 \ddot{y}_2$$

Substituting $\ddot{y}$, $\dot{y}$, and y into equation (1) we have

$$\dot{u}_1 \dot{y}_1 + u_1 \ddot{y}_1 + \dot{u}_2 \dot{y}_2 + u_2 \ddot{y}_2 + p\,u_1 \dot{y}_1 + p\,u_2 \dot{y}_2 + q\,u_1 y_1 + q\,u_2 y_2 = g(t)$$

Rearranging terms we get

$$(\dot{u}_1 \dot{y}_1 + \dot{u}_2 \dot{y}_2) + u_1(\ddot{y}_1 + p\,\dot{y}_1 + q\,y_1) + u_2(\ddot{y}_2 + p\,\dot{y}_2 + q\,y_2) = g(t)$$

The second and third terms on the left-hand side of the equation are equal to zero (why?), which means $(\dot{u}_1 \dot{y}_1 + \dot{u}_2 \dot{y}_2) = g(t)$, and along with assumption (2) we have the system of equations

$$\begin{aligned} \dot{u}_1 y_1 + \dot{u}_2 y_2 &= 0 \\ \dot{u}_1 \dot{y}_1 + \dot{u}_2 \dot{y}_2 &= g(t) \end{aligned} \tag{2}$$

Solving this system, we get $\dot{u}_1 = \frac{-y_2 \dot{u}_2}{y_1}$, and $\dot{u}_2 = \frac{y_1\,g(t)}{y_1 \dot{y}_2 - y_2 \dot{y}_1}$. Substituting $\dot{u}_2$ back into the previous expression we get

$$\dot{u}_1 = \frac{-y_2\,g(t)}{y_1 \dot{y}_2 - y_2 \dot{y}_1} \quad \text{and} \quad \dot{u}_2 = \frac{y_1\,g(t)}{y_1 \dot{y}_2 - y_2 \dot{y}_1}$$

Note that the denominators are simply $W(y_1, y_2)$, which gives us

$$u_1 = \int \frac{-y_2\,g(t)}{W(y_1,y_2)}\,dt \quad \text{and} \quad u_2 = \int \frac{y_1\,g(t)}{W(y_1,y_2)}\,dt$$

Therefore, the solution to the differential equation (1) is

$$y = \underbrace{C_1 y_1 + C_2 y_2}_{\text{complimentary}} + \underbrace{u_1 y_1 + u_2 y_2}_{\text{particular}}$$

For those that took Math 204, equation (2) can be written in matrix form

$$\begin{pmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{pmatrix} \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

Using the inverse matrix we get

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \frac{1}{y_1 \dot{y}_2 - y_2 \dot{y}_1} \begin{pmatrix} \dot{y}_2 & -y_2 \\ -\dot{y}_1 & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

so,

$$\dot{u}_1 = \frac{-y_2\, g(t)}{y_1 \dot{y}_2 - y_2 \dot{y}_1} \text{ and } \dot{u}_2 = \frac{y_1\, g(t)}{y_1 \dot{y}_2 - y_2 \dot{y}_1}$$

Example 1 Find the general solution to $y'' - 2y' + y = e^t \ln(t)$ for $t > 0$.

Example 2 Observe that $y_1(t) = t$ is a solution of the homogeneous equation $t^2 y'' - t y' + y = 0$, $t > 0$. Use this fact so solve the non-homogeneous IVP $t^2 y'' - t y' + y = t$. $y(1) = 1$, $y'(1) = 4$