3.3 Complex Roots of the Characteristic Equation

Example 1 Find the characteristic equation for the differential equation, and possible solutions: $y'' - 4y' + 13y = 0$

Euler's Formula and Complex Numbers

Recall from calculus: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, and $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$. Evaluating e^x with $x = i\,t$ gives

$$\begin{aligned} e^{i\,t} &= \sum_{n=0}^{\infty} \frac{(i\,t)^n}{n!} \\ &= 1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots\right) \\ &= \cos(t) + i\sin(t) \end{aligned}$$

and therefore, $e^{i\mu t} = \cos(\mu t) + i\sin(\mu t)$ ■

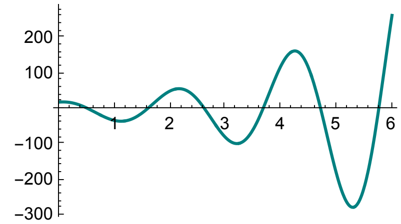
We can now write $e^{(\lambda + i\mu)t}$ in terms of sine and cosine:

$$\begin{aligned} e^{(\lambda + i\mu)t} &= e^{\lambda t} \cdot e^{i\mu t} \\ &= e^{\lambda t}(\cos(\mu t) + i\sin(\mu t)) \\ &= e^{\lambda t}\cos(\mu t) + i\,e^{\lambda t}\sin(\mu t) \end{aligned}$$ ■

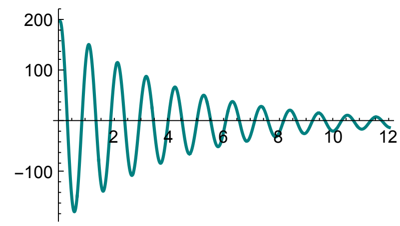
Example 2 Show that the functions $y_1 = e^{\lambda t}\cos(\mu t)$ and $y_2 = e^{\lambda t}\sin(\mu t)$ are linearly independent.

Example 3 Rewrite the general solution for example 1 in terms of sine and cosine, and find the particular solution for $y(0) = 10$ and $y'(0) = 1$.

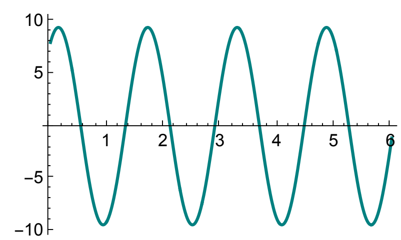
Example 4 $4\,y'' - 4\,y' + 37\,y = 0;\ \ y(0) = 20,\ \ y'(0) = 10;\quad \lambda + \mu\,i = \frac{1}{2} + 3\,i$



Example 5 $2\,y'' + y' + 72.125\,y = 0;\ \ y(0) = 200,\ \ y'(0) = 4;\quad \lambda + \mu\,i = \frac{-1}{4} + 6\,i$



Example 6 $y'' + 16\,y = 0;\ \ y(0) = 8,\ \ y'(0) = 20;\ \ \lambda + \mu\,i = 0 + 4\,i$



Manipulate