3.3 Linear Independence and the Wronskian

Theorem 3.3.3

Let y_1 and y_2 be two solutions to the differential equation $L[y] = y'' + p(t)\,y' + q(t)\,y = 0$ where p and q are continuous on an open interval I. If $W(y_1, y_2)(t) \neq 0$ for all t in I, then y_1 and y_2 are linearly independent

This infers that the following statements are all equivalent:

1. The functions y_1 and y_2 are a fundamental set of solutions on I.
2. The functions are linearly independent on I.
3. $W(y_1, y_2)(t_0) \neq 0$ for some t_0 in I.
4. $W(y_1, y_2)(t) \neq 0$ for all t in I.

Example 1 Determine if the following functions are linearly dependent or linearly independent.

a) $y_1 = e^{-3x}$ and $y_2 = x\,e^{-3x}$

b) $y_1 = \sin(2x)$ and $y = 4\sin(x)\cos(x)$

Abel's Theorem

If y_1 and y_2 are solutions to the differential equation

$$L[y] = y'' \, p(t) \, y'(t) + q(t) \, y = 0$$

where p and q are continuous on an open interval I, then the Wronskian $W[y_1, \, y_2](t)$ is given by

$$W[y_1, \, y_2](t) = c \, e^{-\int p(t) \, dt}$$

where c is a constant then depends of y_1 and y_2 but not t.

Example 2 Find the Wronskian of two solutions of the given differential equation without solving the equation:
$\left(1 - x^2\right) y'' - 2 \, x \, y' + \alpha(\alpha + 1) \, y = 0$. (This is known as Legendre's equation).