

3.2 Fundamental Solutions of Linear Homogeneous Equations; Wronskian

Section 3.1 we looked at how to solve some differential equations in the form

$$a y'' + b y' + c y = 0$$

This section builds on those ideas that can be applied to all second order linear homogeneous equations.

Linear Operators

Let the linear operator L be defined as

$$L[\phi] = \phi'' + p\phi' + q\phi \quad \text{or} \quad L = D^2 + pD + q$$

Example 1 Let $p(t) = t^3$, $q(t) = 2t$, and $\phi(t) = \sin(2t)$, find $L[\phi](t)$.

The Uniqueness and Existence Theorem for First Order Equations

Consider the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

If the functions f and $\frac{\partial f}{\partial y}$ are continuous on some rectangle $\alpha < x < \beta$ and $\gamma < y < \delta$ containing the point (x_0, y_0) , then there exists exactly one solution $y = \phi(x)$ to the initial value problem.

Example 2 Consider the differential equation $x \frac{dy}{dx} = y$. Demonstrate the existence and uniqueness theorem.

The Uniqueness and Existence Theorem for Second Order Equations

Consider the initial value problem

$$y'' + p(t)y' + q(t)y + g(t) = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

where p , q , and g are continuous on an open interval I that contains the point t_0 . Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I .

Example 3 Find the largest interval on which the solution of the initial value problem

$$(t^2 - 5t)y'' + 5y' + ty = 0, \quad y(1) = 0, \quad \text{and} \quad y'(1) = 2$$

is certain to exist.

Theorem: Principal of Superposition

If y_1 and y_2 are two solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

then the linear combination $c_1 y_1 + c_2 y_2$ is also a solution for any values of the constants c_1 and c_2 .

Proof:

Suppose the initial conditions for the equation in the Superposition Principal are $y(t_0) = y_0$ and $y'(t_0) = y'_0$. Then, solving

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y'_1(t_0) + c_2 y'_2(t_0) &= y'_0 \end{aligned}$$

for c_1 and c_2 , we get

$$c_1 = \frac{y_0 y'_2(t_0) - y'_0 y_2(t_0)}{y_1(t_0) y'_2(t_0) - y'_1(t_0) y_2(t_0)} = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}} \quad \text{and similarly} \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}}.$$

The Wronskian

If the determinant $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$, called the Wronskian, of two solutions y_1 and y_2 to the differential equation $L[y] = y'' + p(t)y' + q(t)y = 0$, is not zero on an interval I that contains the point t_0 , then there exists arbitrary constants c_1 and c_2 such that $y = c_1 y_1 + c_2 y_2$ is the family of all solutions to the differential equation.

Example 4 Show that $y_1 = t^2$ and $y_2 = t^{-3/2}$ are two solutions to the differential equation $t^2 y'' + \frac{t}{2} y' - 3y = 0$, and then show that $y = c_1 t^2 + c_2 t^{-3/2}$ is the general solution for all c_1 and c_2 .

Example 5 Verify that the functions $y_1(x) = x$ and $y_2(x) = \sin(x)$ are solutions to the differential equation $(1 - x \cot(x)) y'' - x y' + y = 0$. Do they constitute a fundamental set of solutions?