3.2 Fundamental Solutions of Linear Homogeneous Equations; Wronskian

Section 3.1 we looked at how to solve some differential equations in the form

$$a\,y'' + b\,y' + c\,y = 0$$

This section builds on those ideas that can be applied to all second order linear homogeneous equations.

Linear Operators

Let the linear operator L be defined as

$$L[\phi] = \phi'' + p\,\phi' + q\,\phi \quad \text{or} \quad L = D^2 + p\,D + q$$

Example 1 Let $p(t) = t^3$, $q(t) = 2\,t$, and $\phi(t) = \sin(2\,t)$, find $\mathrm{L}[\phi](\mathrm{t})$.

The Uniqueness and Existence Theorem for First Order Equations

Consider the initial value problem

$$y' = f(x,\, y), \quad y(x_0) = y_0$$

If the functions f and $\frac{\partial f}{\partial y}$ are continuous on some rectangle $\alpha < x < \beta$ and $\gamma < y < \delta$ containing the point $(x_0,\, y_0)$, then there exists exactly one solution $y = \phi(x)$ to the initial value problem.

Example 2 Consider the differential equation $x\,\frac{dy}{dx} = y$. Demonstrate the existence and uniqueness theorem.

The Uniqueness and Existence Theorem for Second Order Equations

Consider the initial value problem

$$y'' + p(t)\,y' + q(t)\,y + g(t) = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

where p, q, and g are continuous on an open interval I that contains the point t_0. Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I.

Example 3 Find the largest interval on which the solution of the initial value problem

$$\left(t^2 - 5\,t\right)y'' + 5\,y' + t\,y = 0, \quad y(1) = 0, \quad \text{and} \quad y'(1) = 2$$

is certain to exist.

Theorem: Principal of Superposition

If y_1 and y_2 are two solutions of of the differential equation

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$

then the linear combination $c_1\,y_1 + c_2\,y_2$ is also a solution for any values of the constants c_1 and c_2.

Proof:

Suppose the initial conditions for the equation in the Superposition Principal are $y(t_0) = y_0$ and $y'(t_0) = y'_0$. Then, solving

$$\begin{aligned} c_1\,y_1(t_0) \;+\; c_2\,y_2(t_0) &= y_0 \\ c_1\,y'_1(t_0) + c_2\,y'_2(t_0) &= y'_0 \end{aligned}$$

for c_1 and c_2, we get

$$c_1 = \frac{y_0\,y'_2(t_0) - y'_0\,y_2(t_0)}{y_1(t_0)\,y'_2(t_0) - y'_1(t_0)\,y_2(t_0)} = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}} \quad \text{and similarly} \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}}.$$

The Wronskian

If the determinant $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$, called the Wronskian, of two solutions y_1 and y_2 to the differential equation $L[y] = y'' + p(t)\,y' + q(t)\,y = 0$, is not zero on an interval I that contains the point t_0, then there exists arbitrary constants c_1 and c_2 such that $y = c_1\,y_1 + c_2\,y_2$ is the family of all solutions to the differential equation.

Example 4 Show that $y_1 = t^2$ and $y = t^{-3/2}$ are two solutions to the differential equation $t^2\,y'' + \frac{t}{2}\,y' - 3\,y = 0$, and then show that $y = c_1\,t^2 + c_2\,t^{-3/2}$ is the general solution for all c_1 and c_2.

Theorem 3.2.3

Let y_1 and y_2 be two solutions to the differential equation $L[y] = y'' + p(t)\,y' + q(t)\,y = 0$ where p and q are continuous on an open interval I. If $W(y_1, y_2)(t) \neq 0$ for all t in I, then y_1 and y_2 are linearly independent.

This infers that the following statements are all equivalent:

1. The functions y_1 and y_2 are a fundamental set of solutions on I.
2. The functions are linearly independent on I.
3. $W(y_1, y_2)(t_0) \neq 0$ for some t_0 in I.
4. $W(y_1, y_2)(t) \neq 0$ for all t in I.

Example 5 Determine if the following functions are linearly dependent or linearly independent.

a) $y_1 = e^{-3x}$ and $y_2 = x\,e^{-3x}$

b) $y_1 = \sin(2\,x)$ and $y = 4\sin(x)\cos(x)$