2.6 Exact Equations and Integrating Factors

Recall the chain rule for functions of single variables: if $y = f(x)$ and $x = g(t)$, then

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

The chain rule for functions of several variables is similar: given $\psi(x, y)$ where $x = f(t)$ and $y = g(t)$, the derivative of $\psi(x, y)$ with respect to t is

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial x}\frac{dx}{dt} + \frac{\partial\psi}{\partial y}\frac{dy}{dt}$$

However, if y is a function of x, then the derivative of $\psi(x, y)$ with respect to x is

$$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x}\frac{dx}{dx} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}$$

Consider the differential equation

$$2\,x\,y + y^3 + \left(x^2 + 3\,x\,y^2\right)\frac{dy}{dx} = 0$$

This equation neither linear or separable. However, notice that the function $\psi(x, y) = x^2\,y + x\,y^3$ has the following partial derivatives:

$$\psi_x = 2\,x\,y + y^3 \quad \text{and} \quad \psi_y = x^2 + 3\,x\,y^2$$

which means we can write the differential equation as

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0$$

or,

$$\frac{d}{dx}\left(x^2\,y + x\,y^3\right) = 0$$

Integrating each side gives $x^2\,y + x\,y^3 = c$, or $\psi(x, y) = c$, which is the implicit solution to the differential equation.

The general form of the exact differential equation can be written as : $M(x, y) + N(x, y)\,\frac{dy}{dx} = 0$, (or in *differential form*: $M(x, y)\,dx + N(x, y)\,dy = 0$) where $M(x, y) = \psi_x$ and $N(x, y) = \psi_y$, and ψ is the solution to the differential equation.

Example 1 Solve the exact differential equation: $3 + \frac{y^3}{x} + \left(3\,y^2\ln(x) + 2\right)\frac{dy}{dx} = 0$.

Example 2 Using the equation in Example 1, find M_y (*e.g.*, ψ_{xy}), and N_x (e.g., ψ_{yx}).

Theorem The differential equation $M(x, y) + N(x, y)\frac{dy}{dx} = 0$ is exact if and only if $M_y(x, y) = N_x(x, y)$.

Example 3 Determine which of the following equations are exact. If it is exact, find the general solution.

a) $(e^x \sin(y) + 2yx) + (e^x \cos(y) + y^2 x)\frac{dy}{dx} = 0$

b) $\left(\frac{y}{x} + 6x\right)dx + (\ln(x) - 2)\,dy = 0$

Non-Exact Equations and Integrating Factors

Non-exact equations may be made exact by multiplying the equation by an integrating factor $\mu(x, y)$, although finding the integrating factor is generally quite difficult. However, if the integrating factor is only in terms of x, or only in terms of y, it is fairly easy to find.

1. If $\xi = \frac{M_y - N_x}{N}$ is a function of x only, the integrating factor is $\mu(x) = e^{\int \xi(x)\,dx}$.
2. If $\xi = \frac{M_y - N_x}{-M}$ is a function of y only, the integrating factor is $\mu(y) = e^{\int \xi(y)\,dy}$.

Example 4 Given the differential equation: $(3xy - y^2) + x(x - y)\frac{dy}{dx} = 0$

a) Show that it is not exact.
b) Find an integrating factor μ.
c) Solve the differential equation.

Example 5 Rewrite the equation in differential form; find an integrating factor, and solve the IVP for $y(1) = 1$: $y' = \frac{y}{e^{-2y} - 2xy}$