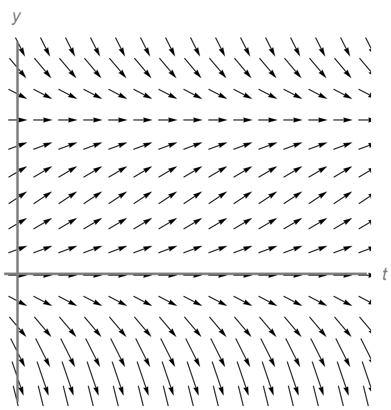
2.5 Autonomous Equations and Population Dynamics

An autonomous equation is first order equation in the form $y'(x) = f(y)$ where the right hand side is only in terms of the unknown function $y(x)$. Two examples of autonomous equation already looked at was unbounded growth $dy/dt = k\,y$, and logistic growth $dy/dt = r\,y\left(1 - \frac{y}{K}\right)$. While these are always separable, they are not always easy to solve. However, we can still get a lot of qualitative information for the solutions without solving them. One important characteristic is the stability, or instability, of the equilibrium solutions.

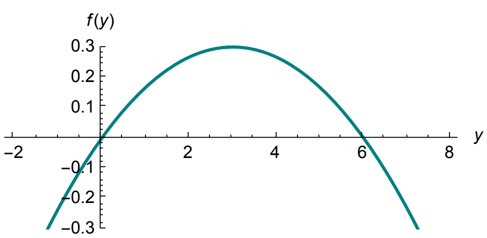
Example 1 Consider the direction field for the logistic growth equation $dy/dt = 0.2\,y\left(1 - \frac{y}{6}\right)$. Find and describe the equilibrium solutions in terms of being stable, unstable, or semi-stable. (For what population value is the growth the maximum?)



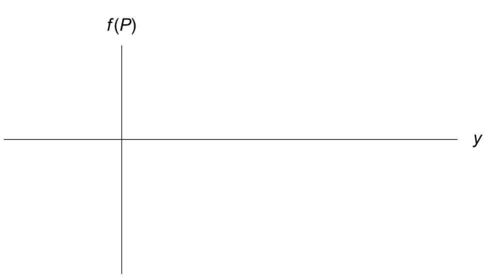
The Phase Line

We can get this same qualitative behavior of the critical points (the equilibrium solutions) by creating the graph of $f(y)$ -vs- y. That is, graph the function $f(y)$ (i.e., y') on the vertical axis and y on the horizontal axis. The vertical axis is called the *phase line* is represented also by the vertical axis in the direction field above.

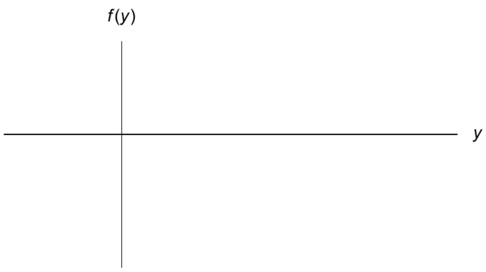
Example 2 Given the plot of $f(y)$ identify the critical points and determine if they are stable, unstable, or semi stable.



Example 3 Determine the stability of the equilibrium points for the equation $dP/dt = -0.5\,P(10 - P)$ which is essentially the logistic formula with $r < 0$. Draw a phase plane and several solution curves.



Example 4 Consider the autonomous equation $dy/dt = -r\,y(1 - y/T)(1 - y/K)$ for positive constants r, T, and K $(T < K)$. Graph the phase line graph, the phase plane, and describe the effect of the parameters K and T.



Example 5 Analyze the equation $dy/dt = -k(1 - y)^2(y - 6)$

Example 6 Analyze the equation $dy/dt = y(3 - y)^2(y + 2)$