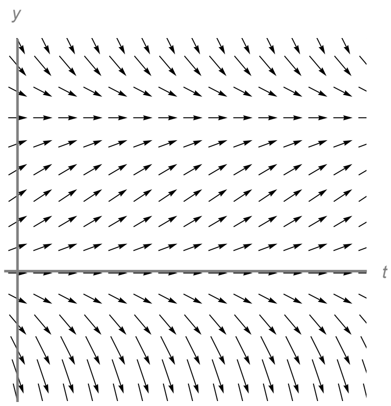
2.5 Autonomous Equations and Population Dynamics

An autonomous equation is first order equation in the form $y'(x) = f(y)$ where the right hand side is only in terms of the unknown function $y(x)$. Two examples of autonomous equation already looked at was unbounded growth $dy/dt = ky$, and logistic growth $dy/dt = r\,y\left(1 - \frac{y}{K}\right)$. While these are always separable, they are not always easy to solve. However, we can still get a lot of qualitative information for the solutions without solving them. One important characteristic is the stability, or instability, of the equilibrium solutions.

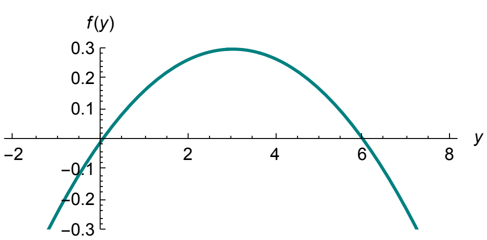
Example 1 Consider the direction field for the logistic growth equation $dy/dt = 0.2\,y\left(1 - \frac{y}{6}\right)$. Find and describe the equilibrium solutions in terms of being stable, unstable, or semi-stable.



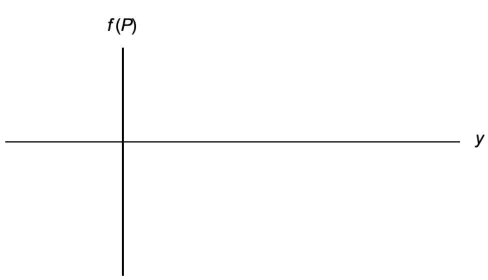
The Phase Line

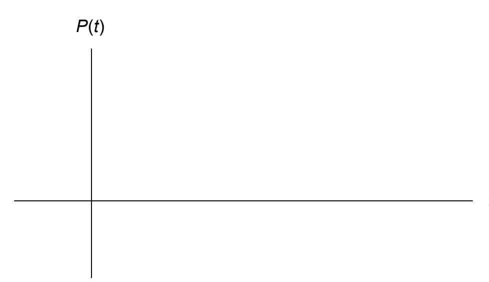
We can get this same qualitative behavior of the critical points (the equilibrium solutions) by creating the graph of $f(y)$ -vs- y. That is, graph the function $f(y)$ (i.e., y') on the vertical axis and y on the horizontal axis. The vertical axis is called the *phase line* is represented also by the vertical axis in the direction field above.

Example 2 Given the plot of $f(y)$ identify the critical points and determine if they are stable, unstable, or semi stable.

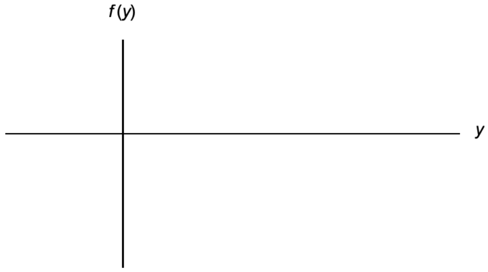


Example 3 Determine the stability of the equilibrium points for the equation $dP/dt = -0.5\,P(10 - P)$ which is essentially the logistic formula with $r < 0$. Draw a phase plane and several solution curves.





Example 4 Consider the autonomous equation $dy/dt = -ry(1 - y/T)(1 - y/K)$ for positive constants r, T, and K ($T < K$). Graph the phase line graph, the phase plane, and describe the effect of the parameters K and T.



Example 5 Analyze the equation $dy/dt = -k(1 - y)^2(y - 6)$

Example 6 Analyze the equation $dy/dt = y(3 - y)^2(y + 2)$