6.1-3 Orthogonality

Inner Product

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$. The **inner product** of **u** and **v** is $(u_1\ u_2\ \cdots\ u_n)\begin{pmatrix} v \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$. This is also called the **dot product** $\mathbf{u} \cdot \mathbf{v}$.

Length of a Vector

The length, or **norm**, of a vector is $\| \mathbf{v} \| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1{}^2 + v_2{}^2 + \cdots v_n{}^2}$. Also note that $\| \mathbf{v} \|^2 = \mathbf{v} \cdot \mathbf{v}$.

A vector whose length is 1 is called a **unit vector** and is found by dividing **v** by its norm $\| \mathbf{v} \|$: $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$. The process of creating **u** from **v** is called **normalizing v**.

Distance Between Two Vectors

For vectors **u** and **v** in $\mathbb{R}^n$, the **distance between u and v**, written as dist(**u**, **v**), is the length of the vector $\mathbf{u} - \mathbf{v}$, that is dist(**u**, **v**) = $\| \mathbf{u} - \mathbf{v} \|$.

Example 1 Find the distance between the vectors $\mathbf{u} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, and draw a sketch of the three vectors **u**, **v**, and $\mathbf{u} - \mathbf{v}$.

Orthogonal Vectors

Recall from pre-calculus that another definition of the dot product is $\mathbf{u} \cdot \mathbf{v} = \| \mathbf{u} \| \, \| \mathbf{v} \| \cos(\theta)$ where θ is the angle between **u** and **v**. If the vectors are in $\mathbb{R}^2$ or $\mathbb{R}^3$ and are perpendicular, then $\theta = 90°$. This means $\mathbf{u} \cdot \mathbf{v} = 0$. In general, two vectors **u** and **v** in $\mathbb{R}^n$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example 2 Determine which vectors are orthogonal: $\mathbf{u}_1 = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

Orthogonal Compliment

Let W be a subspace of $\mathbb{R}^n$. If **z** is orthogonal to every vector in W then **z** is said to be orthogonal to W. The set of all vectors **z** that are orthogonal to W is called the **orthogonal compliment** of W and is denoted by $W^\perp$. This leads to two properties:

(1) A vector **x** is in $W^\perp$ if and only if **x** is orthogonal to every vector in a set that spans W.
(2) $W^\perp$ is a subspace of $\mathbb{R}^n$.

Note: For an $m \times n$ matrix A $(\text{Row } A)^\perp = \text{Nul } A$ and $(\text{Col } A)^\perp = \text{Nul } A^T$. (Thm 6.1.3)

Orthogonal Sets

A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is said to be an orthogonal set if each pair of vectors $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ for all $i \neq j$.

Example 3 Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set, where, $\mathbf{u}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{u}_3 = \begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix}$.

Orthogonal Basis

An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

Theorem

Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination $\mathbf{y} = c_1\,\mathbf{u}_1 + c_2\,\mathbf{u}_2 + \cdots + c_p\,\mathbf{u}_p$ are given by $c_j = \dfrac{\mathbf{y}\cdot\mathbf{u}_j}{\mathbf{u}_j\cdot\mathbf{u}_j}$, $(j = 1, \ldots, p)$.

Example 4 The set $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of vectors in example 3 form an orthogonal basis for $\mathbb{R}^3$. Express the vector $\mathbf{y} = \begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix}$ in terms of S.

Orthonormal Sets; Orthonormal Basis

An orthogonal set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ each of unit length is called an **orthonormal set**. If W is a subspace spanned by an orthonormal set, then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis.

Example 5 Give an example of two well known orthonormal bases.

Theorem: An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

Orthogonal Projection

Recall from calculus the quantity $c = \dfrac{\mathbf{y}\cdot\mathbf{u}}{\|\boldsymbol{u}\|}$ is the scalar projection of $\mathbf{y}$ onto $\mathbf{u}$, or the magnitude (amount) of $\mathbf{y}$ in the direction of $\mathbf{u}$. The **orthogonal projection** is the vector $\hat{\mathbf{y}} = c\,\mathbf{u}$, or $\hat{\mathbf{y}} = \dfrac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\,\mathbf{u}$. If L is the subspace spanned by $\mathbf{u}$ then $\hat{\boldsymbol{y}} = \mathrm{proj}_L\,\boldsymbol{y}$.

Example 6 Find the closest approximation to $\mathbf{y} = \langle 3, 1, 5, 1\rangle$ spanned by $\mathbf{v}_1 = \langle 3,1,-1,1\rangle$ and $\mathbf{v}_2 = \langle 1, -1, 1, -1\rangle$