5.6 Discrete Dynamical Systems

Consider the simplistic interaction between the population of foxes and rabbits in a forest, and how they change monthly. In the absence of foxes, the population of rabbits will grow rapidly, and the presence of foxes will decrease the rabbit population. However, in the absence of rabbits the fox population will decrease, and with a substantial rabbit population the fox population will increase. This is an example of a *dynamical system*. Possible recurrence equations for the respective populations are

$$R_{k+1} = 1.1\,R_k - 0.104\,F_k$$
$$F_{k+1} = 0.4\,R_k + 0.5\,F_k$$

This system can be written as $\boldsymbol{x}_{k+1} = A\,\boldsymbol{x}_k$ where $\boldsymbol{x}_k = \begin{pmatrix} R_k \\ F_k \end{pmatrix}$ and $A = \begin{pmatrix} 1.1 & -0.104 \\ 0.4 & 0.5 \end{pmatrix}$. Using an initial population vector $\boldsymbol{x}_0$ we can create a Markov chain for the population for rabbits and foxes each month.

Recall, if A is an $n \times n$ matrix, and $\mathbf{x}_{k+1} = A\,\mathbf{x}_k$, we can write $\mathbf{x}_k$ in terms of the eigenvalues λ_1 and λ_2, and the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ provided that the eigenvectors are linearly independent:

$$\mathbf{x}_k = c_1\,{\lambda_1}^k\,\mathbf{v}_1 + c_2\,{\lambda_2}^k\,\mathbf{v}_2$$

Analyzing the above equation in terms of the eigenvalues can shed light on the dynamical system.

Example 1 Given $A = \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix}$ and $\mathbf{x}_0 = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$, and let $\mathbf{x}_{k+1} = A\,\mathbf{x}_k$. Use the following TI-84 program find graph $\mathbf{x}_k$ for $k = 0, 1, 2, 3, \ldots, 10$. Then, find a formula for x_k in terms of the eigensystem and calculate the long-term behavior.

TI-84 DYNAMIC Program

```
[B]→[C]
Prompt N
Pt-on([C](1,1),[C](2,1))
For(I,1,N)
[A][C]→[C]
Pt-on([C](1,1),[C](2,1))
End
```

Enter the stochastic matrix in [A], the initial vector in [B] and set the window to $x[-8, 8]$ and $y[-8, 8]$. Run the program choosing N between 10 and 50. The final vector is stored into [C]. To clear the graph press 2nd DRAW and ClrDraw.

Complex Eigenvalues with Magnitude 1

Example 2 Let $A = \begin{pmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{pmatrix}$ and $x_0 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$. Use DYNAMIC to plot $\mathbf{x}_k$ for $k = 0, \ldots, 50$.

An eigenvalue of A is $\lambda = 0.8 - 0.6\,i$ and the corresponding eigenvector is $\boldsymbol{v} = \begin{pmatrix} -2 - 4\,i \\ 5 \end{pmatrix}$. Let $P = (\text{Re}\,\mathbf{v} \;\; \text{Im}\,\mathbf{v})$. Show that $C = P^{-1}\,A\,P$ is the rotation matrix "inside" A.

The Origin as an Attractor, Repellor, or Saddle Point

Example 3 For the following matrices A and initial vector $x_0 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ use DYNAMIC to plot several points in the window $x[-8, 8]$ and $y[-8, 8]$ to find the behavior of the system with respect to the origin. Classify the origin as a **repellor**, **attractor**, or a **saddle point**. Find a relation between the behavior and the eigenvalues of A.

(a) $\begin{pmatrix} 0.8 & 0 \\ 0 & 0.7 \end{pmatrix}$ (b) $\begin{pmatrix} 1.2 & 0 \\ 0 & 1.4 \end{pmatrix}$ (c) $\begin{pmatrix} 0.5 & 0 \\ 0 & 1.2 \end{pmatrix}$

Complex Eigenvalues with Magnitude $\neq 1$

Example 4 Find the trajectory of points using matrix A and $\mathbf{x}_0$ from example (2) except

(a) Change 0.5 to 0.4 (b) Change 0.5 to 0.6.

How do the trajectories relate to the magnitude of the eigenvalues?

Answers