# 5.5 Complex Eigenvalues and Eigenvectors

**Example 1 a** Find the characteristic polynomial for the matrix $A = \begin{pmatrix} 1 & -5 \\ 2 & 7 \end{pmatrix}$, and the eigenvalues.

**Example 1 b** Using one eigenvalue set up the system of equations $(A - \lambda_1 I_2)\,\mathbf{x}_1 = \mathbf{0}$, and solve for $\mathbf{x}_1$. Note, the two equations in the system *both* give relations for the unique eigenvector $\mathbf{x}_1$. You only need to use one of them to actually find the eigenvector.

**Example 1 c** Find the second eigenvector for $\lambda_2$.

**Example 2** Find the eigenvalues and eigenvectors for the matrix $\begin{pmatrix} 4 & -5 \\ 2 & -2 \end{pmatrix}$.

Matrices with complex eigenvalues and eigenvectors are an important transformation. Find the eigenvalues of the following matrix transformations:

a) $\begin{pmatrix} k & 0 \\ 0 & h \end{pmatrix}$ b) $\begin{pmatrix} k & 1 \\ 0 & h \end{pmatrix}$ c) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ d) $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$

**Example 3** Show that a matrix in the form $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ has eigenvalues $\lambda_{1,2} = a \pm b\,i$, and is a composition of a scaling matrix and a rotation matrix, where $r = \sqrt{a^2 + b^2}$ is the scaling factor, and $\phi$ is the rotation angle.

We can now find the scaling and rotation "hidden" in a matrix $A$ that has complex eigenvalues and eigenvectors.

**Example 4** Choose the eigenvalue with a negative imaginary part (i.e., $a - b\,i$) and the associated eigenvector from example 2, and let $P = [\text{Re}\,\mathbf{v} \quad \text{Im}\,\mathbf{v}]$.

a) Find the matrix $C = P^{-1} A P$ and compare it with the matrix in example 3.

b) Since the $|\lambda| \neq 1$ matrix $C$ is not a pure rotation. Divide each element in $C$ by $|\lambda|$ and multiply on the outside by $|\lambda|$.

c) Identify the rotation and the scaling produced by matrix $A$.

*(Example 5.5.3 Manipulate)*

### Theorem 9

Let $A$ be a real 2×2 matrix with a complex eigenvalue $\lambda = a - b\,i$, $(b \neq 0)$, and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$. Then

$$A = P\,C\,P^{-1}, \quad \text{where} \quad P = [\text{Re}\,\mathbf{v} \quad \text{Im}\,\mathbf{v}] \quad \text{and} \quad C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

**Example 5** Find the scaling and rotation for the matrix $A = \begin{pmatrix} 3 & -5 \\ 2 & 5 \end{pmatrix}$.