5.3 Diagonalization

Suppose we need to evaluate A^k for large values of k, a common operation in several applications in linear Algebra. Eigenvalues and eigenvectors can help speed up the process.

Recall, to find $\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}^4$ we need to compute $\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 17 & -28 \\ -56 & 129 \end{pmatrix}$. Very tedious.

Example 1 Given $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ find, A^2, A^3, and A^k.

Diagonalizing a Matrix

A matrix is **diagonalizable** if there exists an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

Example 2 Given $A = \begin{pmatrix} 7 & -15 \\ 2 & -4 \end{pmatrix}$, and $P = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$, show that A is diagonalize to $D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, and find a formula for A^k.

Theorem 5: The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n independent eigenvectors. In fact, $A = PDP^{-1}$, if and only if the columns of P are linearly independent eigenvectors of A. In this case, the diagonal entries of D are the eigenvalues of A that correspond to the eigenvectors in P respectively.

Example 3 Diagonalize the matrix: $A = \begin{pmatrix} 4 & 9 \\ 3 & -2 \end{pmatrix}$ and find A^{10}

Example 4 Diagonalize the matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 0 & 1 \\ 9 & -6 & 5 \end{pmatrix}$. Be sure to check the eigenvectors form a linearly independent set. (If they don't form a basis for $\mathbb{R}^3$, A is not diagonalizable.)

Example 5 True/False: If an $n \times n$ matrix has n distinct eigenvalues then A is automatically diagonalizable? (Theorem 6)