# 5.2 The Characteristic Equation

**Example 1** Use the equation $A\mathbf{x} = \lambda\mathbf{x}$ to find the eigenvalues for the matrix $\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$. Use the fact from the IMT that for a homogeneous equation an invertible matrix has only the trivial solution $\mathbf{x} = \mathbf{0}$, also meaning a non trivial solution must have a non-invertible matrix.

### The Invertible Matrix Theorem Continued

Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if:

(s) The number 0 is *not* an eigenvalue of $A$

(t) The determinant of $A$ is *not* 0.

## The Characteristic Equation

A scalar $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$ if and only if $\lambda$ satisfies the characteristic equation

$$\det(A - \lambda I) = 0$$

**Example 2** Use the characteristic polynomial to find the eigenvalues for $\begin{pmatrix} 3 & 8 \\ 5 & 6 \end{pmatrix}$, and the eigenvectors.

**Example 3** Find the characteristic polynomial and eigenvalues for $\begin{pmatrix} 3 & 2 & 0 & 3 \\ 0 & -2 & 5 & 8 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 4 \end{pmatrix}$.

## Similar Matrices and Similarity

If $A$ and $B$ are $n \times n$ matrices, then ***A* is similar to *B*** if there is an invertible matrix $P$ such that $P^{-1} A P = B$, or equivalently $A = P B P^{-1}$. If $Q = P^{-1}$ then the second equation becomes $Q^{-1} B Q = A$, so $B$ is also similar to $A$. Changing $A$ into $P^{-1} A P$ is called a **similarity transformation**.

**Example 4** Show that $A = \begin{pmatrix} 1 & 1 \\ -2 & 5 \end{pmatrix}$ is similar to $B = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$ with $P = \begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix}$.

### Theorem 4

If $n \times n$ matrices $A$ and $B$ are similar, then they have the same characteristic polynomial and therefore the same eigenvalues.

## Application of Eigenvalues and Eigenvectors to Markov Chains

**Example 5**

Recall the Urban-Rural migration stochastic matrix from 4.9 notes, $P = \begin{pmatrix} 0.95 & 0.08 \\ 0.05 & 0.92 \end{pmatrix}$ with an initial population vector $x_0 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$. We can use the eigenvalues and eigenvectors to solve for any $\mathbf{x}_k$ and the steady state vector. First, find the eigenvalues and eigenvectors using *Mathematica*:

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Eigensystem[{{95 / 100, 8 / 100}, {5 / 100, 92 / 100}}]
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$$\left\{\left\{1, \frac{87}{100}\right\}, \left\{\left\{\frac{8}{5}, 1\right\}, \{-1, 1\}\right\}\right\}$$

Next, write the initial state $\mathbf{x}_0 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$ in terms of the eigenvectors: $\mathbf{x}_0 = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = (\mathbf{v}_1 \; \mathbf{v}_2) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, or: $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = (\mathbf{v}_1 \; \mathbf{v}_2)^{-1} \mathbf{x}_0$

Use the fact that with the eigenvalues and eigenvectors we have $P \mathbf{v}_1 = 1 \mathbf{v}_1$ and $P \mathbf{v}_2 = 0.87 \mathbf{v}_2$. Compute $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_k$ in terms of the eigenvalues and eigenvectors.

Find the steady state $\mathbf{x}_\infty$, i.e., $\mathbf{x}_k$ as $k \to \infty$.