5.1 Eigenvalues, Eigenvectors, and Eigenspaces

Example 1 Consider the matrix $A = \begin{pmatrix} 3 & 2 \\ 5 & 6 \end{pmatrix}$, the vector $\mathbf{u} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and the scalar $\lambda = 8$.

(a) Find $A\mathbf{u}$ (b) Find $\lambda\mathbf{u}$

Definition

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. The scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.

Example 2 Given $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is also an eigenvector of A from Example 1, find the corresponding eigenvalue.

Example 3 The matrix $\begin{pmatrix} 2 & 5 & 4 \\ 3 & 1 & -1 \\ 1 & 2 & 3 \end{pmatrix}$ has an eigenvalue of -2. Find the corresponding eigenvector by solving $A\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x}$.

Example 4 Show that multiplying the eigenvector by a scalar still results in an eigenvector.

This means the set of all solutions to the equation $A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$ or $(A - \lambda I)\mathbf{x} = \mathbf{0}$ is just the null space of $A - \lambda I$, and is therefore a subspace of $\mathbb{R}^n$ called the **eigenspace** of A corresponding to λ.

Example 5 Find the eigenspace of $A = \begin{pmatrix} 3 & 2 & 5 \\ 4 & 1 & 5 \\ 4 & 2 & 4 \end{pmatrix}$ for the eigenvalue $\lambda_1 = -1$. Show that another eigenvalue is 10, and find its eigenvector.

Example 6 Given the *triangular* matrix $\begin{pmatrix} 3 & 2 & -4 \\ 0 & 1 & 8 \\ 0 & 0 & 2 \end{pmatrix}$ show that the eigenvalues are the entries along the main diagonal. What does it mean if an eigenvalue is 0?

Theorem 1

The eigenvalues of a triangular matrix (either upper or lower) are the entries along the main diagonal.

Example 7 Determine if the eigenvectors of Example 5 are linearly dependent or independent.

Theorem 2

If $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an $n \times n$ matrix A, then the set of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ is linearly independent.

Mathematica Command

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In[18]:= Eigensystem[{{3, 2, 5}, {4, 1, 5}, {4, 2, 4}}]
Out[18]= {{10, -1, -1}, {{1, 1, 1}, {-5, 0, 4}, {-1, 2, 0}}}
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