4.9 Application: Markov Chains and Stochastic Matrices

Over the last hundred years people have migrated from the rural areas to the urban areas. Let u_k represent the *percentage* of the population in the urban area, and v_k be the percentage in the rural areas. Also, suppose 5% of the population moves from urban to rural (95% stay) and 8% move from rural to urban (92% stay) yearly. Finally, suppose at a point in time the rural population (initial population) was 15 000 and the urban population was 10 000.

a) Find an expression to calculate the P_{k+1} population percentages, and write it as a transition matrix equation.
b) Calculate the population percentage after $k = 1, 2,$ and 3 years.
c) Use your calculator to estimate the long-term population percentages. This is called a **steady-state** vector.
d) Analytically calculate the steady-state vector and compare the answer to part (c).

Definition

(1) A column vector with elements that sum to 1 is called a **probability** vector.

(2) A stochastic matrix P is called **regular** if there exists a k such that P^k contains strictly positive entries.

Example 1 Show that $P = \begin{pmatrix} 0.2 & 0.75 & 0 \\ 0.8 & 0 & 0.1 \\ 0 & 0.25 & 0.9 \end{pmatrix}$ is a regular matrix.

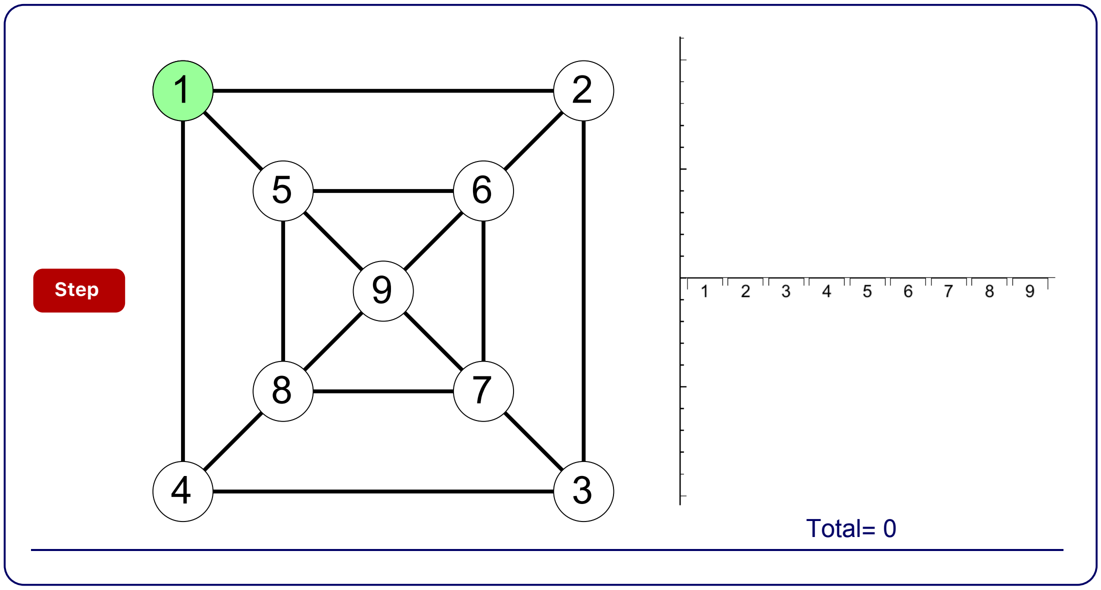
Theorem

If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector $\mathbf{q}$. Further, if $\mathbf{x}_0$ is any initial state and $\mathbf{x}_{k+1} = P\,\mathbf{x}_k$ for $k = 0, 1, 2, \ldots$, then the Markov chain $\{\mathbf{x}_k\}$ converges to $\mathbf{q}$ as $k \to \infty$.

Example 2 Given the stochastic matrix $A = \begin{pmatrix} 0.3 & 0.1 & 0.2 \\ 0.4 & 0.9 & 0.3 \\ 0.3 & 0 & 0.5 \end{pmatrix}$, approximate the steady-state probability vector with your calculator, and calculate the steady-state vector directly solving $A\,\mathbf{x} = \mathbf{x}$.

Markov Pinball

Example 3 Suppose you have a pinball machine with nine places, or states, and a ball can visit states randomly as shown below:



The probability the ball will move from one state to the next are:

1: 30%⟶2, 70%⟶4
2: 80%⟶1, 20%⟶6
3: 70%⟶2, 30%⟶4
4: 90%⟶3, 10%⟶8
5: 20%⟶1, 60%⟶6, 20%⟶9
6: 80%⟶7, 20%⟶9
7: 20%⟶3, 60%⟶8, 20%⟶9
8: 20%⟶4, 80%⟶5
9: 20%⟶6, 20%⟶8, 60%⟶9

a) Create a probability matrix for the movement of the pinball using a **left stochastic matrix**, i.e., $P\mathbf{x}_0 = \mathbf{x}_1$.
b) If a ball starts at position 7, find the probability it will be at position 5 after 4 moves.
c) Find the steady state vector for the position of the pinball assuming any initial state.

Cat and Mouse

Example 4 Suppose you have five consecutive boxes in a row with a cat in the first box and a mouse in the fifth box. At the end of each minute both the cat and the mouse **randomly** jump to an adjacent box. If the cat and mouse ever occupy the same box the cat eats the mouse and the "game" is over. We want to know, on average, how long does the game last? (E.g., how long does the mouse live?)

a) Write down each possible state as an ordered pair (cat, mouse).

b) Find the **right stochastic matrix** for a transition (meaning the state vector is on the left and the stochastic matrix is on the right.)

c) Given the initial state for the cat and mouse $\boldsymbol{\tau}_0 = [1, 0, 0, 0, 0]$, i.e. boxes (1, 5), find the probability state vector after 2 minutes (or two jumps), that is $\boldsymbol{\tau}_2 = \boldsymbol{\tau}_0 T^2$.

d) Show $\boldsymbol{\tau}\left(I + T + T^2 + T^3 + \cdots\right) = \boldsymbol{\tau}(I - T)^{-1}$ and use this to find the average number of steps the game lasts for each possible initial state. Note: we don't need to include state 5 since being in that state doesn't contribute anything to the mouses longevity.

Simple Code: