4.7 Change of Base

Suppose we have one coordinate system with basis $\mathcal{B}$, and a second coordinate system with basis $\mathcal{C}$. Given a point with $\mathcal{B}$ coordinates, $[\mathbf{x}]_{\mathcal{B}}$ how can you find the corresponding $\mathcal{C}$ coordinates? One method would be to find the coordinates in the standard basis, $[\mathbf{x}]_{\mathcal{E}}$, and then convert these in to $\mathcal{C}$ coordinates.

Example 1 Suppose we have two coordinate systems: one with basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, and a second with basis $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\} = \left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}\right\}$.

a) Given $\mathbf{b}_1 = \mathbf{c}_1 + \mathbf{c}_2$, and $\mathbf{b}_2 = 2\,\mathbf{c}_1 - \mathbf{c}_2$, find the basis $\mathcal{B}$.

b) If the coordinates of a point in $\mathcal{B}$ are $[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ find the corresponding coordinates in the standard basis, i.e., $[\mathbf{x}]_{\mathcal{E}} = \mathbf{x}$.

c) Using the $\mathcal{E}$-coordinates, $\mathbf{x}$, found in part (b) find the corresponding points in $\mathcal{C}$-coordinates.

An important part in the above conversion was knowing the $\mathcal{C}$ coordinates for the $\mathcal{B}$ basis vectors. Knowing these we can find a matrix that converts directly from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates, i.e., $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

Example 2 Find the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ that transforms the coordinates $[\mathbf{x}]_{\mathcal{B}}$ directly into $[\mathbf{x}]_{\mathcal{C}}$

Notice that the entries of matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ found in Example 2 are the $\mathcal{C}$-coefficients for the basis $\mathcal{B}$ vectors. This means the conversion matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ can be found by solving the equations

$$[\mathbf{c}_1 \;\; \mathbf{c}_2]\,\mathbf{x} = \mathbf{b}_1 \quad \text{and} \quad [\mathbf{c}_1 \;\; \mathbf{c}_2]\,\mathbf{x} = \mathbf{b}_2$$

These two equations can be solved simultaneously giving

$$[\mathbf{c}_1 \;\; \mathbf{c}_2 \;\; | \;\; \mathbf{b}_1 \;\; \mathbf{b}_2] \sim \left[\, I_n \;\middle|\; \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \,\right]$$

Example 3 Given the basis $\mathcal{B}=\left\{\begin{pmatrix}4\\2\end{pmatrix},\begin{pmatrix}-2\\4\end{pmatrix}\right\}$ and $C=\left\{\begin{pmatrix}3\\-1\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix}\right\}$ find $\underset{C\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{B}\leftarrow C}{P}$. Test using the point $\left[\begin{pmatrix}2\\1\end{pmatrix}\right]_{\mathcal{B}}$. Make a sketch of each coordinate system and the point.

Example 4 Suppose $f_1 = 2x^2 + x - 3$, $f_2 = x^2$, and $f_3 = x + 4$; and $g_1 = x^2 + 2x$, $g_2 = x^2 + 1$, and $g_3 = 1 - x$. Find a linear combination of g_i polynomials that gives the same polynomial $h = 2f_1 - 3f_2 + f_3$.