4.6 Row Space; Rank

Suppose you have a 20×50 matrix with random real elements and you need to know how many linearly independent columns there are and how many linearly independent rows there are. Will they be equal, or always different? Hmmm.

Row Space

The row space of an $m \times n$ matrix A is the set of all possible linear combinations of the row vectors of A. Since each row vector has n elements, Row A is a subspace of $\mathbb{R}^n$. A 4×6 matrix will have Row $A = \{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4\}$, where each vector is in $\mathbb{R}^6$. Also, using the Spanning Set Theorem, we can remove any dependent row vectors from the spanning set. But how can the dependent row vectors be identified?

Theorem 13

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for B.

Example 1 Find a basis for the row space, column space, and null space for matrix A. Assume A and B are row equivalent.

$$A = \begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{pmatrix} \sim B = \begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition: Rank

The **rank** of a matrix A is the dimension of the column space. Also, since Row A is the same as Col A^T, the dimension of the row space of A is the rank of A^T.

Theorem 14: The Rank Theorem

The dimensions of the column space and row space of an $m \times n$ matrix A are equal, and is the **rank** of A. The rank of A is equal to the number of pivot position in A and satisfies the equation

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n$$

Example 2 a) If a 5×7 matrix has a null space of three dimensions, what is the rank of A.

b) A student found a 4×7 matrix has a two dimensional null space. Is this possible?

Example 3 An engineer solving a homogeneous system of 8 equations in 11 variables found three independent solutions, and all other solutions are linear combinations of these three. Can they be certain of a solution to the non-homogeneous equation $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$?

The Invertible Matrix Theorem Continued

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix:

m. The columns of A form a basis of $\mathbb{R}^n$.
n. $\operatorname{Col} A = \mathbb{R}^n$
o. $\dim \operatorname{Col} A = n$
p. $\operatorname{rank} A = n$
q. $\operatorname{Nul} A = \{\mathbf{0}\}$
r. $\dim \operatorname{Nul} A = 0$