4.5 The Dimension of a Vector Space

Theorem 9

If a vector space V has basis $\mathcal{B} = \{\mathbf{b}_{1,} \ldots, \mathbf{b}_n\}$, then any set in V containing more that n vectors must be linearly dependent.

Theorem 10

If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Definition

If V is spanned by a finite set of vectors, then V is said to be **finite-dimensional**, and the **dimension** of V, written as dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

Example 1 What is the dimension of the standard basis for the space of polynomials $\mathbb{P}_3$?

Example 2 What is the dimension of the subspace $H = \left\{ \begin{pmatrix} 2a - b + 2c \\ d \\ 3a + 2b - 4c \\ b - 2c + 3d \end{pmatrix} : a,\ b,\ c,\ d \in \mathbb{R} \right\}$?

Example 3 Find the subspaces of $\mathbb{R}^3$, and interpret the subspaces geometrically.

Theorem 11

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite dimensional and $\dim H \le \dim V$.

Theorem 12 The Basis Theorem

Let V be a p dimensional vector space, $p \ge 1$. Any linearly independent set of exactly p elements is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

Example 4 Find the dimensions of the null space and column space of A and relate the dimensions to the number of pivot positions and free variables.

$$A = \begin{pmatrix} 1 & 3 & -2 & -2 & 4 \\ 2 & 1 & 0 & -1 & 1 \\ -3 & 0 & -1 & 2 & 1 \end{pmatrix}$$

Example 5 Laguerre Polynomials are solutions to the Laguerre differential equation $x\,y'' + (1-x)\,y' + n\,y = 0$ where n is a non-negative integer. The first four Laguerre polynomials for $n = 0, 1, 2, 3$ are: $\left\{1,\ 1-x,\ \frac{1}{2}(2-4x+x^2),\ \frac{1}{6}(6-18x+9x^2-x^3)\right\}$. Show that these polynomials form a basis for $\mathbb{P}_3$ and find the dimension of the basis. Also, find the **p** coordinates for the polynomial $3x^3 + 4x^2 - 8$.