4.4 Coordinate Systems

Recall the standard basis for $\mathbb{R}^3$ is $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. It's often advantageous to use a different basis which also spans $\mathbb{R}^3$. One theorem and a definition allows us to convert between bases.

Theorem 7 The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$, such that

$$\mathbf{x} = c_1\,\mathbf{b}_1, \ldots, c_n\,\mathbf{b}_n$$

Definition

Suppose $\mathcal{B} = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis $\boldsymbol{\mathcal{B}}$** (or the $\mathcal{B}$-coordinates of $\mathbf{x}$) are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\,\mathbf{b}_1, \ldots c_n\,\mathbf{b}_n$.

Example 1 Show that $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$. For the vector $[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ in $\mathbb{R}^3$, find the coordinates of $\mathbf{x}$ in $\mathcal{E}$.

Example 2 Using the basis $\mathcal{B}$ in Example 1, find the $[\mathbf{x}]_{\mathcal{B}}$ coordinates for the vector $\mathbf{x} = \begin{pmatrix} -2 \\ 9 \\ -6 \end{pmatrix}$. The matrix that changes $\mathcal{B}$-coordinates to $\mathcal{E}$-coordinates is often denoted as $P_{\mathcal{B}}$. That is, $P_{\mathcal{B}} = [\mathbf{b}_1 \;\; \mathbf{b}_2 \; \ldots \; \mathbf{b}_n]$. Therefore, $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$.

Graphical Representation of Coordinates in $\mathbb{R}^2$

Example 3 Suppose $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$. Draw a coordinate system for $\mathcal{B}$ and plot the $[\mathbf{x}]_{\mathcal{B}}$ points (−1, 1), (2, 1), (2, 2), and (−1, 2). Next, draw a coordinate system for $\mathcal{E}$ (the standard *x*-y coordinate system); find and plot the points **x.**

Example 4 Let $\mathcal{B}$ be the standard basis of the space $\mathbb{P}_2$. Show that the polynomials $3t+4$, t^2-2t+1, and $2t^2-t+6$ are linearly dependent by writing them as the vectors: $\mathbf{p} \mapsto [\mathbf{p}]_{\mathcal{B}}$.

Example 4 was an example of an *isomorphism*; iso meaning *same,* and morphism meaning *shape*. The mapping in Example 4 denoted as $\mathbf{p} \mapsto [\mathbf{p}]_{\mathcal{B}}$, i.e., $a_0 + a_1 t + a_2 t^2 = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$, is an isomorphism from $\mathbb{P}_2$ onto $\mathbb{R}^3$. All vector-space operations in $\mathbb{P}_2$ correspond to operations in $\mathbb{R}^3$.