4.3 Linear Independent Sets; Bases

Linear Independence

Recall that a set of vectors is linearly independent if the trivial solution is the only solution to $c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 + \cdots + c_n \mathbf{v}_n = \mathbf{0}$. This is good when we have n-tuple vectors, but breaks down when the vector space is more generalized.

Theorem 4

An indexed set $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ $(j > 1)$ is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{j-1}$.

Example 1 Consider the vector space of polynomials $\mathbf{p}_1 = t$, $\mathbf{p}_2 = 1 + 2t$, and $\mathbf{p}_3 = 4t + 3$. Is the set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ linearly dependent or independent?

Example 2 Consider the set $H = \{\cos(x), \sin(x)\}$. Is this set linearly dependent or independent in the set of all continuous function of the Interval $[0, 2\pi]$, i.e., $C[0, 2\pi]$?

Bases

Definition

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_p\}$ in V is **basis** for H if

a. $\mathcal{B}$ is a linearly independent set, and
b. the subspace spanned by $\mathcal{B}$ coincides with H; that is

$$H = \text{Span}\{\mathbf{b}_1, \cdots, \mathbf{b}_p\}$$

Example 3 Do the columns of an invertible matrix A in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$?

Note: in physics and calculus (among other sciences) the **standard basis** in $\mathbb{R}^n$ is $\{\mathbf{e}_1, \mathbf{e}_2, \cdots \mathbf{e}_n\}$ where $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$.

Example 4 Do the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ form a basis for $\mathbb{R}^3$?

Notice the basis in Example 4 is not as efficient as the standard basis for $\mathbb{R}^3$ as $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Theorem 5: The Spanning Set Theorem

Let $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ be a set in V, and let $H = \text{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$.

a. If one of the vectors in S is a linear combination of the remaining vectors in S, then the set formed from S by removing that vector still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

Example 5 Consider the subspace spanned by $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -6 \\ 3 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$. Show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ equals $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, meaning a spanning set for H is $\{\mathbf{v}_1, \mathbf{v}_2\}$.

Bases for the Null Space and Column Space

For a matrix A, a basis for the column space is the set of linearly independent column vectors of A. That is, the original column vectors associated with the pivot positions. The basis for the null space is found as before.

Example 6 Find a basis for the column space and the null space for the matrix $A = \begin{pmatrix} 2 & 4 & 1 & 3 & 3 \\ 2 & 4 & 0 & 8 & 4 \\ 1 & 2 & 1 & -1 & 1 \end{pmatrix}$.