4.1 Vector Spaces and Subspaces

Everything we've covered so far can be generalized into more formal concepts called **vector spaces** and **subspaces**. Keep in mind, much of the new idea here is simply *terminology* applied to previous ideas. This will be true for the next several sections.

Definition

A **vector space** is a non-empty set of objects, called vectors, on which are two defined operations called *addition*, and *multiplication by scalars*.

Axioms of a vector space. Let **u**, **v**, and **w** be vectors in a vector space V, and c and d be scalars.

1. The sum of **u** and **v**, denoted $\mathbf{u} + \mathbf{v}$, is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a zero vector **0** in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each **u** in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. The scalar multiple of **u** by c, denoted by $c\,\mathbf{u}$, is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\,\mathbf{u} + c\,\mathbf{v}$.
8. $(c + d)\,\mathbf{u} = c\,\mathbf{u} + d\,\mathbf{u}$.
9. $c(d\,\mathbf{u}) = (c\,d)\,\mathbf{u}$.
10. $1\,\mathbf{u} = \mathbf{u}$

Examples of Vector Spaces

1. $\mathbb{R}^n$: points in n-dimensions.
2. Forces used in physics.
3. The set of second order linear differential equations.
4. $\mathbb{P}_n$: the set of polynomials of degree n.

Subspaces

A subspace is a subset of vectors from a larger vector space. However, $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$, since any point on a plane in $\mathbb{R}^3$ still has three coordinates, and hence in $\mathbb{R}^3$.

To show that a **subspace** is a subspace of a larger vector space, only three of the 10 axioms need to be checked (the rest are automatically verified.)

Definition of a Subspace

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.
b. H is closed under addition.
c. H is closed under scalar multiplication.

Example 1 Is the set consisting of only the zero vector in the vector space V a subspace of V?

Example 2 Let W be the union of the first and third quadrants in the x-y plane. That is $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x\,y \geq 0 \right\}$. If **u** is in W and c is any scalar, is c**u** in W. Explain. Is it possible to find a **u** and **v** in W such that **u**+**v** is not in W? What does this mean?

Example 3 A plane through the origin is a subspace of $\mathbb{R}^3$, but a plane not through the origin is not a subspace of $\mathbb{R}^3$. Explain.

Spanning Sets: A Subspace Spanned by a Set

Theorem

If $v_1, v_2 \cdots v_p$ are in a vector space V, then $\text{Span}\{v_1, v_2 \cdots v_p\}$ is a subspace of V.

Example 4 Let H be the set if vectors of the form $(2\,a,\ a + 3\,b,\ b)$ where a and b are scalars. Show that H is a subspace of $\mathbb{R}^3$.