3.3 Cramer's Rule

Cramer's Rule is a method for solving and analyzing the matrix equation $A\mathbf{x} = \mathbf{b}$, although it's not very efficient when A is larger than 2×2.

Suppose A is an $n \times n$ matrix, and $A\mathbf{x} = \mathbf{b}$ can be written $[a_1 \; a_2 \; a_3 \cdots a_i \cdots a_{n-1} \; a_n]\,\mathbf{x} = \mathbf{b}$. Let $A_i(\mathbf{b})$ be the matrix obtained by replacing the i^{th} column with $\mathbf{b}$.

Theorem 7 Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$\mathbf{x}_i = \frac{\det(A_i(\mathbf{b}))}{\det(A)}, \qquad i = 1, 2, \cdots, n$$

Example 1 Use Cramer's rule to solve the system: $\begin{cases} 3x - 4y = 8 \\ 2x + 5y = 2 \end{cases}$

Example 2 Use any method to solve $\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases}$, and rewrite it in Cramer's form.

Example 3 Use Cramer's Rule to analyze the solutions to the system $\begin{cases} 2cx + 6y = 3 \\ 12x + cy = 2 \end{cases}$.

Using Cramer's Rule To Find an Inverse

The adjugate of a matrix A, $\operatorname{adj} A$, is defined as the transpose of the cofactors C_{ij}, where the cofactor $C_{ij} = (-1)^{i+j} A_{ij}$. The inverse can be computed as: $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$.

Example 4 Find the adjugate of $A = \begin{pmatrix} 2 & 3 & 5 \\ -2 & 1 & 6 \\ 3 & 2 & 8 \end{pmatrix}$, and $\operatorname{adj}(A) \cdot A$, then find A^{-1}.

Area and Volume

Theorem 9

If A is a 2×2 matrix, the area of a parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Example 5 Calculate the volume of the parallelepiped with corner points (–5, 4, 2), (–3, 6, 1), (1, –1, 4), and (3, 1, 2).

Example 6 Find the absolute value of the determinant for each transformation matrix, and describe it's significance in terms of the area of a transformed region.

a. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ b. $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ c. $\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$ d. $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ e. $\begin{pmatrix} k & 0 \\ 0 & h \end{pmatrix}$

Theorem 10

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a finite region in $\mathbb{R}^2$, then

$$\{\text{Area of } T(S)\} = |\det A| \cdot \{\text{Area of } S\}$$

Similarly, if $T : \mathbb{R}^3 \to \mathbb{R}^3$ and S is a finite volume in $\mathbb{R}^3$, then

$$\{\text{Volume of } T(S)\} = |\det A| \cdot \{\text{Volume of } S\}$$

Example 7 Find the area of the ellipse $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$ by considering it is the transformed circle $x^2 + y^2 = 1$.