3.1 Introduction to Determinants

Recall the determinant of a 2×2 matrix is $D=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=a\,d-c\,b$, and if $D\neq0$ the matrix is non-singular, i.e., the matrix has an inverse. Similarly, we need to be able to find the determinant of an $n\times n$ matrix where $n>2$. First, a few necessary pieces:

Submatrices and Cofactor Expansion

Suppose we have a $n\times n$ matrix and cover up the row and column containing the e_{ij} element. The remaining elements make an $(n-1)\times(n-1)$ **submatrix**. For example the submatrix for the element for $\begin{pmatrix} 2 & 3 & -1 \\ 6 & -3 & 5 \\ 2 & 9 & 3 \end{pmatrix}$ resulting by covering up element $e_{23}=5$ is $\begin{pmatrix} 2 & 3 \\ 2 & 9 \end{pmatrix}$. Also, when finding the determinant of a matrix larger than 2×2 we need to keep in mind the "checker-board sign" matrix: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$. This is a matrix where the sign of the ij[th] element is $(-1)^{i+j}$. Finally, the determinant of 3×3 matrix is calculated by using the elements in the first row as coefficients multiplying the determinant of their respective submatrix, and adding or subtracting determined by the "checker-board matrix".

The determinant of the matrix above is:

$$\begin{vmatrix} 2 & 3 & -1 \\ 6 & -3 & 5 \\ 2 & 9 & 3 \end{vmatrix}=2\begin{vmatrix} -3 & 5 \\ 9 & 3 \end{vmatrix}-3\begin{vmatrix} 6 & 5 \\ 2 & 3 \end{vmatrix}+(-1)\begin{vmatrix} 6 & -3 \\ 2 & 9 \end{vmatrix}=2(-54)-3(8)-60=-192$$

This is called the **cofactor expansion** method to find the determinant of a matrix.

Theorem 1

The determinant of an $n\times n$ matrix A can be computed by a cofactor expansion across any row or any column.

Example 1 Use cofactor expansion to find the determinant of $A=\begin{pmatrix} 2 & -1 & 5 \\ 2 & 4 & -5 \\ 4 & 3 & 8 \end{pmatrix}$ using any row or column.

Example 2 How many 2×2 determinants would be required to compute to find the determinant of a 6×6 matrix? An Intel Core i7 at 3.9 GHz can do roughly 127,273 MIPS (million instructions per second.) How long would it take a computer to compute the determinant for a 25×25 matrix using cofactor expansion.

Example 3 Row reduce the matrix $\begin{pmatrix} 2 & -1 & 5 \\ 2 & 4 & -5 \\ 4 & 3 & 8 \end{pmatrix}$ into echelon form (not rref) and find the product of the main diagonal.

Theorem 2

If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.

Example 4 Demonstrate theorem 2 for a general 4×4 triangular matrix.

Example 5 Find the determinant of the elementary transformation matrices in $\mathbb{R}^3$ for (a) scaling a row by k, (b) interchanging two rows, and (b) adding a multiple k of one row to another row.

Example 6 Calculate the area of the parallelogram formed by the vectors $\mathbf{u} = \langle 7, 0 \rangle$ and $\mathbf{v} = \langle 2, 5 \rangle$. Find the determinant of the matrix $A = [\mathbf{u}\ \ \mathbf{v}]$.